

# THE DYNAMICS OF THE 3D RADIAL NLS WITH THE COMBINED TERMS

CHANGXING MIAO, GUIXIANG XU, AND LIFENG ZHAO

ABSTRACT. In this paper, we show the scattering and blow-up result of the radial solution with the energy below the threshold for the nonlinear Schrödinger equation (NLS) with the combined terms

$$iu_t + \Delta u = -|u|^4 u + |u|^2 u \quad (\text{CNLS})$$

in the energy space  $H^1(\mathbb{R}^3)$ . The threshold is given by the ground state  $W$  for the energy-critical NLS:  $iu_t + \Delta u = -|u|^4 u$ . This problem was proposed by Tao, Visan and Zhang in [37]. The main difficulty is the lack of the scaling invariance. Illuminated by [17], we need give the new radial profile decomposition with the scaling parameter, then apply it into the scattering theory. Our result shows that the defocusing,  $\dot{H}^1$ -subcritical perturbation  $|u|^2 u$  does not affect the determination of the threshold of the scattering solution of (CNLS) in the energy space.

## 1. INTRODUCTION

We consider the dynamics of the radial solutions for the nonlinear Schrödinger equation (NLS) with the combined nonlinearities in  $H^1(\mathbb{R}^3)$

$$\begin{cases} iu_t + \Delta u = f_1(u) + f_2(u), & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0(x) \in H^1(\mathbb{R}^3). \end{cases} \quad (1.1)$$

where  $u : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{C}$  and  $f_1(u) = -|u|^4 u$ ,  $f_2(u) = |u|^2 u$ . As we known,  $f_1$  has the  $\dot{H}^1$ -critical growth,  $f_2$  has the  $\dot{H}^1$ -subcritical growth.

The equation has the following mass and Hamiltonian quantities

$$M(u)(t) = \frac{1}{2} \int_{\mathbb{R}^3} |u(t, x)|^2 dx; \quad E(u)(t) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u(t, x)|^2 dx + F_1(u(t)) + F_2(u(t))$$

where  $F_1(u(t)) = -\frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 dx$ ,  $F_2(u(t)) = \frac{1}{4} \int_{\mathbb{R}^3} |u(t, x)|^4 dx$ . They are conserved for the sufficient smooth solutions of (1.1).

---

2000 *Mathematics Subject Classification.* Primary: 35L70, Secondary: 35Q55.

*Key words and phrases.* Blow up; Dynamics; Nonlinear Schrödinger Equation; Scattering; Threshold Energy.

In [37], Tao, Visan and Zhang made the comprehensive study of

$$iu_t + \Delta u = |u|^4 u + |u|^2 u$$

in the energy space. They made use of the interaction Morawetz estimate established in [6] and the stability theory for the scattering solution. Their result is based on the scattering result of the defocusing, energy-critical NLS in the energy space, which is established by Bourgain [3, 4] for the radial case, I-team [7], Ryckman-Visan [34] and Visan [38] for the general data. Since the classical interaction Morawetz estimate in [6] fails for (1.1), Tao, et al., leave the scattering and blow-up dichotomy of (1.1) below the threshold as an open problem in [37]. For other results, please refer to [15, 16, 30, 31, 32, 39, 40].

For the focusing, energy-critical NLS

$$iu_t + \Delta u = -|u|^4 u. \quad (1.2)$$

Kenig and Merle first applied the concentration compactness in [2, 21, 22] into the scattering theory of the radial solution of (1.2) in [19] with the energy below that of the ground state of

$$-\Delta W = |W|^4 W. \quad (1.3)$$

In this paper, we will also make use of the concentration compactness argument and the stability theory to study the dichotomy of the radial solution of (1.1) with the energy below the threshold, which will be shown to be the energy of the ground state  $W$  for (1.2). For the applications of the concentration compactness in the scattering theory and rigidity theory of the critical NLS, NLW, NLKG and Hartree equations, please see [8, 9, 10, 11, 12, 13, 17, 20, 23, 24, 25, 26, 27, 28, 29].

We now show the differences between (1.1) and (1.2). On one hand, there is an explicit solution  $W$  for (1.2), which is the ground state of (1.3) and does not scatter. The threshold of the scattering solution of (1.2) is determined by the energy of  $W$ . While for (1.1), there is no such explicit solution, whose energy is the threshold of the scattering solution of (1.1). We need look for a mechanism to determine the threshold of the scattering solution of (1.1). It turns out that the constrained minimization of the energy as (1.5) is appropriate<sup>1</sup>. On the other hand, for (1.2), it is  $\dot{H}^1$ -scaling invariant, which gives us many conveniences, especially in the nonlinear profile decomposition about (1.2). While for (1.1), it is the lack of scaling invariance. We need give the new

---

<sup>1</sup>The similar constrained minimization of the energy as (1.5) is not appropriate for the focusing perturbation:  $iu_t + \Delta u = -|u|^4 u - |u|^2 u$ , since the threshold  $m$  in this way equals to 0 and it is not the desired result.

profile decomposition with the scaling parameter of (1.1) in  $H^1(\mathbb{R}^3)$ , take care of the role of the scaling parameter in the linear and nonlinear profile decompositions, then apply them into the scattering theory.

Now for  $\varphi \in H^1$ , we denote the scaling quantity  $\varphi_{3,-2}^\lambda$  by

$$\varphi_{3,-2}^\lambda(x) = e^{3\lambda}\varphi(e^{2\lambda}x).$$

We denote the scaling derivative of  $E$  by  $K(\varphi)$

$$K(\varphi) = \mathcal{L}E(\varphi) := \frac{d}{d\lambda}\Big|_{\lambda=0} E(\varphi_{3,-2}^\lambda) = \int_{\mathbb{R}^3} \left( \frac{4}{2}|\nabla\varphi|^2 - \frac{12}{6}|\varphi|^6 + \frac{6}{4}|\varphi|^4 \right) dx, \quad (1.4)$$

which is connected with the Virial identity, and then plays the important role in the blow-up and scattering of the solution of (1.1).

Now the threshold  $m$  is determined by the following constrained minimization<sup>2</sup> of the energy  $E(\varphi)$

$$m = \inf\{E(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) = 0\}. \quad (1.5)$$

Since we consider the  $\dot{H}^1$ -critical growth with the  $\dot{H}^1$ -subcritical perturbation, we will use the modified energy later

$$E^c(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2}|\nabla u(t, x)|^2 - \frac{1}{6}|u(t, x)|^6 \right) dx.$$

As the nonlinearity  $|u|^2u$  is the defocusing,  $\dot{H}^1$ -subcritical perturbation, one think that the focusing,  $\dot{H}^1$ -critical term plays the decisive role of the threshold of the scattering solution of (1.1) in the energy space. The first result is to characterize the threshold energy  $m$  as following

**Proposition 1.1.** *There is no minimizer for (1.5). But for the threshold energy  $m$ , we have*

$$m = E^c(W),$$

where  $W \in \dot{H}^1(\mathbb{R}^3)$  is the ground state of the massless equation

$$-\Delta W = |W|^4 W.$$

As the dynamics of the solution of (1.1) with the energy less than the threshold  $m$ , the conjecture is

---

<sup>2</sup>In fact, the following minimization of the static energy

$$\inf\{M(\varphi) + E(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) = 0\}$$

also equals to  $m$ .

**Conjecture 1.2.** *Let  $u_0 \in H^1(\mathbb{R}^3)$  with*

$$E(u_0) < m, \tag{1.6}$$

*and  $u$  be the solution of (1.1) and  $I$  be its maximal interval of existence. Then*

- (a) *If  $K(u_0) \geq 0$ , then  $I = \mathbb{R}$ , and  $u$  scatters in both time directions as  $t \rightarrow \pm\infty$  in  $H^1$ ;*
- (b) *If  $K(u_0) < 0$ , then  $u$  blows up both forward and backward at finite time in  $H^1$ .*

In this paper, we verify the conjecture in the radial case.

**Theorem 1.3.** *Conjecture 1.2 holds whenever  $u$  is spherically symmetric.*

**Remark 1.4.** Our consideration of the radial case is based on the following facts:

- (1) It is an open problem that the scattering result of (1.2) in dimension three, except for the radial case in [19]. Our result is based on the corresponding scattering result of (1.2).
- (2) It seems to be hard to lower the regularity of the critical element to  $L^\infty \dot{H}^s$  for some  $s < 0$  by the double Duhamel argument in dimension three to obtain the compactness of the critical element in  $L^2$ , which is used to control the spatial center function  $x(t)$  of the critical element.

**Remark 1.5.** We can remove the radial assumption under the stronger constraint that

$$M(u_0) + E(u_0) < m,$$

which can help us to obtain the compactness of the critical element in  $L^2$  and control the spatial center function  $x(t)$  of the critical element. Of course, we need the precondition<sup>3</sup> that the global wellposedness and scattering result of (1.2) holds for  $u_0 \in \dot{H}^1(\mathbb{R}^3)$  with

$$\int_{\mathbb{R}^3} \left( |\nabla u_0|^2 - |u_0|^6 \right) dx \geq 0, \quad \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_0|^2 - \frac{1}{6} |u_0|^6 \right) dx < m.$$

---

<sup>3</sup>By the relation between the sharp Sobolev constant and the ground state  $W$ , we know that the constrained condition

$$\int_{\mathbb{R}^3} \left( |\nabla u_0|^2 - |u_0|^6 \right) dx \geq 0, \quad \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_0|^2 - \frac{1}{6} |u_0|^6 \right) dx < E^c(W)$$

is equivalent to the constrained condition

$$\|\nabla u_0\|_{L^2}^2 \leq \|\nabla W\|_{L^2}^2, \quad \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_0|^2 - \frac{1}{6} |u_0|^6 \right) dx < E^c(W).$$

We use the former in this paper while the latter is given by Kenig-Merle in [19].

**Remark 1.6.** From the assumption in Theorem 1.3, we know that the solution starts from the following subsets of the energy space,

$$\begin{aligned}\mathcal{K}^+ &= \left\{ \varphi \in H^1(\mathbb{R}^3) \mid \varphi \text{ is radial, } E(\varphi) < m, K(\varphi) \geq 0 \right\}, \\ \mathcal{K}^- &= \left\{ \varphi \in H^1(\mathbb{R}^3) \mid \varphi \text{ is radial, } E(\varphi) < m, K(\varphi) < 0 \right\}.\end{aligned}$$

By the scaling argument, we know that  $\mathcal{K}^\pm \neq \emptyset$  (we can also know that  $\mathcal{K}^+ \neq \emptyset$  by the small data theory). In fact, let  $\chi(x)$  be a radial smooth cut-off function satisfying  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ . If we take  $\chi_R(x) = \chi(x/R)$  and

$$\varphi(x) = \theta \lambda^{-1/2} \chi_R(x/\lambda) W(x/\lambda),$$

where  $\theta, \lambda, R$  is determined later and the cutoff function  $\chi_R$  is not needed for dimension  $d \geq 5$  since  $W \in H^1$ . Then we have

$$\begin{aligned}\|\nabla \varphi\|_{L^2}^2 &= \theta^2 \left( \|\nabla W\|_{L^2}^2 + \int \left( (\chi_R^2 - 1) |\nabla W|^2 + |\nabla \chi_R|^2 |W|^2 + 2\chi_R \nabla \chi_R \cdot W \nabla W \right) dx \right), \\ \|\varphi\|_{L^6}^6 &= \theta^6 \left( \|W\|_{L^6}^6 + \int (\chi_R^6 - 1) |W|^6 dx \right), \quad \|\varphi\|_{L^4}^4 = \lambda \cdot \theta^4 \|\chi_R W\|_{L^4}^4, \\ \|\varphi\|_{L^2}^2 &= \lambda^2 \cdot \theta^2 \|\chi_R W\|_{L^2}^2.\end{aligned}$$

Therefore, taking  $R$  sufficiently large,  $\theta = 1 + \epsilon$  and  $\lambda = \epsilon^3$ , we have

$$\begin{aligned}E(\varphi) &= \frac{\theta^2}{2} \|\nabla W\|_{L^2}^2 - \frac{\theta^6}{6} \|W\|_{L^6}^6 \\ &\quad + \frac{\theta^2}{2} \int \left( (\chi_R^2 - 1) |\nabla W|^2 + |\nabla \chi_R|^2 |W|^2 + 2\chi_R \nabla \chi_R \cdot W \nabla W \right) dx \\ &\quad - \frac{\theta^6}{6} \int (\chi_R^6 - 1) |W|^6 dx + \lambda \cdot \frac{\theta^4}{4} \|\chi_R W\|_{L^4}^4 \\ &= m - 6\epsilon^2 m + o(\epsilon^2), \\ K(\varphi) &= 2\theta^2 \|\nabla W\|_{L^2}^2 - 2\theta^6 \|W\|_{L^6}^6 \\ &\quad + 2\theta^2 \int \left( (\chi_R^2 - 1) |\nabla W|^2 + |\nabla \chi_R|^2 |W|^2 + 2\chi_R \nabla \chi_R \cdot W \nabla W \right) dx \\ &\quad - 2\theta^6 \int (\chi_R^6 - 1) |W|^6 dx + \lambda \cdot \frac{3\theta^4}{2} \|\chi_R W\|_{L^4}^4 \\ &= -24\epsilon m + o(\epsilon^2).\end{aligned}$$

If taking  $\epsilon < 0$  and  $|\epsilon|$  sufficient small, then we have  $\varphi \in \mathcal{K}^+$ ; If taking  $\epsilon > 0$  and sufficient small, then we have  $\varphi \in \mathcal{K}^-$ .

**Acknowledgements.** The authors are partly supported by the NSF of China (No. 10801015, No. 10901148, No. 11171033). The authors would like to thank Professor K. Nakanishi for his valuable communications.  $\square$

## 2. PRELIMINARIES

In this section, we give some notation and some wellknown results.

**2.1. Littlewood-Paley decomposition and Besov space.** Let  $\Lambda_0(x) \in \mathcal{S}(\mathbb{R}^3)$  such that its Fourier transform  $\tilde{\Lambda}_0(\xi) = 1$  for  $|\xi| \leq 1$  and  $\tilde{\Lambda}_0(\xi) = 0$  for  $|\xi| \geq 2$ . Then we define  $\Lambda_k(x)$  for any  $k \in \mathbb{Z} \setminus \{0\}$  and  $\Lambda_{(0)}(x)$  by the Fourier transforms:

$$\tilde{\Lambda}_k(\xi) = \tilde{\Lambda}_0(2^{-k}\xi) - \tilde{\Lambda}_0(2^{-k+1}\xi), \quad \tilde{\Lambda}_{(0)}(\xi) = \tilde{\Lambda}_0(\xi) - \tilde{\Lambda}_0(2\xi).$$

Let  $s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$ . The inhomogeneous Besov space  $B_{p,q}^s$  is defined by

$$B_{p,q}^s = \left\{ f \mid f \in \mathcal{S}', \left\| 2^{ks} \|\Lambda_k * f\|_{L_x^p} \right\|_{l_{k \geq 0}^q} < \infty \right\},$$

where  $\mathcal{S}'$  denotes the space of tempered distributions. The homogeneous Besov space  $\dot{B}_{p,q}^s$  can be defined by

$$\dot{B}_{p,q}^s = \left\{ f \mid f \in \mathcal{S}', \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} 2^{qks} \|\Lambda_k * f\|_{L_x^p}^q + \|\Lambda_{(0)} * f\|_{L_x^p}^q \right)^{1/q} < \infty \right\}.$$

**2.2. Linear estimates.** We say that a pair of exponents  $(q, r)$  is Schrödinger  $\dot{H}^s$ -admissible in dimension three if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s$$

and  $2 \leq q, r \leq \infty$ . If  $I \times \mathbb{R}^3$  is a space-time slab, we define the  $\dot{S}^0(I \times \mathbb{R}^3)$  Strichartz norm by

$$\|u\|_{\dot{S}^0(I \times \mathbb{R}^3)} := \sup \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^3)}$$

where the sup is taken over all  $L^2$ -admissible pairs  $(q, r)$ . We define the  $\dot{S}^s(I \times \mathbb{R}^3)$  Strichartz norm to be

$$\|u\|_{\dot{S}^s(I \times \mathbb{R}^3)} := \|D^s u\|_{\dot{S}^0(I \times \mathbb{R}^3)}.$$

We also use  $\dot{N}^0(I \times \mathbb{R}^3)$  to denote the dual space of  $\dot{S}^0(I \times \mathbb{R}^3)$  and

$$\dot{N}^k(I \times \mathbb{R}^3) := \{u; D^k u \in \dot{N}^0(I \times \mathbb{R}^3)\}.$$

By definition and Sobolev's inequality, we have

**Lemma 2.1.** *For any  $\dot{S}^1$  function  $u$  on  $I \times \mathbb{R}^3$ , we have*

$$\|\nabla u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}(I \times \mathbb{R}^3)} + \|u\|_{L_t^\infty L_x^6} + \|u\|_{L_t^{12} L_x^9} + \|u\|_{L_{t,x}^{10}} \lesssim \|u\|_{\dot{S}^1}.$$

*For any  $\dot{S}^{1/2}$  function  $u$  on  $I \times \mathbb{R}^3$ , we have*

$$\|u\|_{L_t^\infty \dot{H}_x^{1/2}} + \|u\|_{L_t^6 \dot{B}_{18/7,2}^{1/2}(I \times \mathbb{R}^3)} + \|u\|_{L_t^\infty L_x^3} + \|u\|_{L_t^6 L_x^{9/2}} + \|u\|_{L_{t,x}^5} \lesssim \|u\|_{\dot{S}^{1/2}}.$$

Now we state the standard Strichartz estimate.

**Lemma 2.2** ([5, 18, 36]). *Let  $I$  be a compact time interval,  $k \in [0, 1]$ , and let  $u : I \times \mathbb{R}^3 \rightarrow \mathbb{C}$  be an  $\dot{S}^k$  solution to the forced Schrödinger equation*

$$iu_t + \Delta u = F$$

*for a function  $F$ . Then we have*

$$\|u\|_{\dot{S}^k(I \times \mathbb{R}^3)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^d)} + \|F\|_{\dot{N}^k(I \times \mathbb{R}^3)},$$

*for any time  $t_0 \in I$ .*

We shall also need the following exotic Strichartz estimate, which is important in the application of the stability theory.

**Lemma 2.3** ([14]). *For any  $F \in L_t^2 \left( I; \dot{B}_{18/11,2}^{1/3} \right)$ , we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}} \lesssim \|F\|_{L_t^2 \dot{B}_{18/11,2}^{1/3}}.$$

**2.3. Local wellposedness and Virial identity.** Let

$$ST(I) := L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9 \cap L_t^6 \dot{B}_{18/7,2}^{1/2} \cap L_{t,x}^5(I \times \mathbb{R}^3).$$

By the definition of admissible pair, we know that  $L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9$  is the  $\dot{H}^1$ -admissible space,  $L_t^6 \dot{B}_{18/7,2}^{1/2} \cap L_{t,x}^5$  is the  $\dot{H}^{1/2}$ -admissible space. Now we have

**Theorem 2.4** ([37]). *Let  $u_0 \in H^1$ , then for every  $\eta > 0$ , there exists  $T = T(\eta)$  such that if*

$$\|e^{it\Delta} u_0\|_{ST([-T, T])} \leq \eta,$$

*then (1.1) admits a unique strong  $H_x^1$ -solution  $u$  defined on  $[-T, T]$ . Let  $(-T_{\min}, T_{\max})$  be the maximal time interval on which  $u$  is well-defined. Then,  $u \in S^1(I \times \mathbb{R}^d)$  for every compact time interval  $I \subset (-T_{\min}, T_{\max})$  and the following properties hold:*

(1) *If  $T_{\max} < \infty$ , then*

$$\|u\|_{ST((0, T_{\max}) \times \mathbb{R}^d)} = \infty.$$

Similarly, if  $T_{min} < \infty$ , then

$$\|u\|_{ST((-T_{min}, 0) \times \mathbb{R}^d)} = \infty.$$

- (2) The solution  $u$  depends continuously on the initial data  $u_0$  in the following sense: The functions  $T_{min}$  and  $T_{max}$  are lower semicontinuous from  $\dot{H}_x^1 \cap \dot{H}_x^{1/2}$  to  $(0, +\infty]$ . Moreover, if  $u_0^{(m)} \rightarrow u_0$  in  $\dot{H}_x^1 \cap \dot{H}_x^{1/2}$  and  $u^{(m)}$  is the maximal solution to (1.1) with initial data  $u_0^{(m)}$ , then  $u^{(m)} \rightarrow u$  in  $ST(I \times \mathbb{R}^3)$  and every compact subinterval  $I \subset (-T_{min}, T_{max})$ .

*Proof.* The proof is based on the Strichartz estimate and exotic Strichartz estimate and the following nonlinear estimates.

$$\begin{aligned} \| |u|^4 u \|_{L_t^2 \dot{B}_{18/11,2}^{1/3}} &\lesssim \|u\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}} \|u\|_{L_{t,x}^{10}}^4, \quad \| |u|^2 u \|_{L_t^2 \dot{B}_{18/11,2}^{1/3}} \lesssim \|u\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}} \|u\|_{L_{t,x}^5}^2, \\ \| |u|^4 u \|_{L_t^2 \dot{B}_{6/5,2}^{1/2}} &\lesssim \|u\|_{L_t^6 \dot{B}_{18/7,2}^{1/2}} \|u\|_{L_t^{12} L_x^9}^4, \quad \| |u|^2 u \|_{L_t^2 \dot{B}_{6/5,2}^{1/3}} \lesssim \|u\|_{L_t^6 \dot{B}_{18/7,2}^{1/2}} \|u\|_{L_t^6 L_x^{9/2}}^2. \end{aligned}$$

□

**Lemma 2.5.** Let  $\phi \in C_0^\infty(\mathbb{R}^3)$ , radially symmetric and  $u$  be the radial solution of (1.1). Then we have

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3} \phi(x) |u(t, x)|^2 dx &= -2\Im \int_{\mathbb{R}^3} \nabla \phi \cdot \nabla \bar{u} u dx \\ \partial_t^2 \int_{\mathbb{R}^3} \phi(x) |u(t, x)|^2 dx &= 4 \int_{\mathbb{R}^3} \phi''(r) |\nabla u|^2 dx - \int_{\mathbb{R}^3} \Delta^2 \phi |u(t, x)|^2 dx \\ &\quad - \frac{4}{3} \int_{\mathbb{R}^3} \Delta \phi |u(t, x)|^6 dx + \int_{\mathbb{R}^3} \Delta \phi |u(t, x)|^4 dx, \end{aligned}$$

where  $r = |x|$ .

*Proof.* By the simple computation, we have

$$\begin{aligned} \partial_t^2 \int_{\mathbb{R}^3} \phi(x) |u(t, x)|^2 dx &= 4 \int_{\mathbb{R}^3} \phi_{jk} \cdot \Re(u_k \bar{u}_j) dx - \int_{\mathbb{R}^3} \Delta^2 \phi \cdot |u(t, x)|^2 dx \\ &\quad - \frac{4}{3} \int_{\mathbb{R}^3} \Delta \phi \cdot |u(t, x)|^6 dx + \int_{\mathbb{R}^3} \Delta \phi \cdot |u(t, x)|^4 dx. \end{aligned}$$

Then the result comes from the following fact

$$\partial_{jk}^2 \phi(x) = \phi''(r) \frac{x_j x_k}{r^2} + \frac{\phi'(r)}{r} \left( \delta_{jk} - \frac{x_j x_k}{r^2} \right)$$

holds for any radial symmetric function  $\phi(x)$ . □

**2.4. Variational characterization.** In this subsection, we give the threshold energy  $m$  (Proposition 1.1) by the variational method, and various estimates for the solutions



of (1.1) with the energy below the threshold. There is no the radial assumption on the solution.

We first give some notation before we show the behavior of  $K$  near the origin. Let us denote the quadratic and nonlinear parts of  $K$  by  $K^Q$  and  $K^N$ , that is,

$$K(\varphi) = K^Q(\varphi) + K^N(\varphi),$$

where  $K^Q(\varphi) = 2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx$ , and  $K^N(\varphi) = \int_{\mathbb{R}^3} \left( -2|\varphi|^6 + \frac{3}{2}|\varphi|^4 \right) dx$ .

**Lemma 2.6.** *For any  $\varphi \in H^1(\mathbb{R}^3)$ , we have*

$$\lim_{\lambda \rightarrow -\infty} K^Q(\varphi_{3,-2}^\lambda) = 0. \quad (2.1)$$

*Proof.* It is obvious by the definition of  $K^Q$ .  $\square$

Now we show the positivity of  $K$  near 0 in the energy space.

**Lemma 2.7.** *For any bounded sequence  $\varphi_n \in H^1(\mathbb{R}^3) \setminus \{0\}$  with*

$$\lim_{n \rightarrow +\infty} K^Q(\varphi_n) = 0,$$

*then for large  $n$ , we have*

$$K(\varphi_n) > 0.$$

*Proof.* By the fact that  $K^Q(\varphi_n) \rightarrow 0$ , we know that  $\lim_{n \rightarrow +\infty} \|\nabla \varphi_n\|_{L^2}^2 = 0$ . Then by the Sobolev and Gagliardo-Nirenberg inequalities, we have for large  $n$

$$\begin{aligned} \|\varphi_n\|_{L_x^6}^6 &\lesssim \|\nabla \varphi_n\|_{L_x^2}^6 = o(\|\nabla \varphi_n\|_{L^2}^2), \\ \|\varphi_n\|_{L_x^4}^4 &\lesssim \|\varphi_n\|_{L^2} \|\nabla \varphi_n\|_{L^2}^3 = o(\|\nabla \varphi_n\|_{L^2}^2), \end{aligned}$$

where we use the boundedness of  $\|\varphi_n\|_{L^2}$ . Hence for large  $n$ , we have

$$K(\varphi_n) = \int_{\mathbb{R}^3} \left( 2|\nabla \varphi_n|^2 - 2|\varphi_n|^6 + \frac{3}{2}|\varphi_n|^4 \right) dx \approx \int_{\mathbb{R}^3} |\nabla \varphi_n|^2 dx > 0.$$

This concludes the proof.  $\square$

By the definition of  $K$ , we denote two real numbers by

$$\bar{\mu} = \max\{4, 0, 6\} = 6, \quad \underline{\mu} = \min\{4, 0, 6\} = 0.$$

Next, we show the behavior of the scaling derivative functional  $K$ .

**Lemma 2.8.** *For any  $\varphi \in H^1$ , we have*

$$(\bar{\mu} - \mathcal{L}) E(\varphi) = \int_{\mathbb{R}^3} \left( |\nabla \varphi|^2 + |\varphi|^6 \right) dx,$$

$$\mathcal{L}(\bar{\mu} - \mathcal{L})E(\varphi) = \int_{\mathbb{R}^3} \left(4|\nabla\varphi|^2 + 12|\varphi|^6\right) dx.$$

*Proof.* By the definition of  $\mathcal{L}$ , we have

$$\mathcal{L}\|\nabla\varphi\|_{L^2}^2 = 4\|\nabla\varphi\|_{L^2}^2, \quad \mathcal{L}\|\varphi\|_{L^6}^6 = 12\|\varphi\|_{L^6}^6, \quad \mathcal{L}\|\varphi\|_{L^4}^4 = 6\|\varphi\|_{L^4}^4,$$

which implies that

$$(\bar{\mu} - \mathcal{L})E(\varphi) = 6E(\varphi) - K(\varphi) = \int_{\mathbb{R}^3} \left(|\nabla\varphi|^2 + |\varphi|^6\right) dx,$$

$$\mathcal{L}(\bar{\mu} - \mathcal{L})E(\varphi) = \mathcal{L}\|\nabla\varphi\|_{L^2}^2 + \mathcal{L}\|\varphi\|_{L^6}^6 = \int_{\mathbb{R}^3} \left(4|\nabla\varphi|^2 + 12|\varphi|^6\right) dx.$$

This completes the proof.  $\square$

According to the above analysis, we will replace the functional  $E$  in (1.5) with a positive functional  $H$ , while extending the minimizing region from “ $K(\varphi) = 0$ ” to “ $K(\varphi) \leq 0$ ”. Let

$$H(\varphi) := \left(1 - \frac{\mathcal{L}}{\bar{\mu}}\right) E(\varphi) = \int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla\varphi|^2 + \frac{1}{6}|\varphi|^6\right) dx,$$

then for any  $\varphi \in H^1 \setminus \{0\}$ , we have

$$H(\varphi) > 0, \quad \mathcal{L}H(\varphi) \geq 0.$$

Now we can characterization the minimization problem (1.5) by use of  $H$ .

**Lemma 2.9.** *For the minimization  $m$  in (1.5), we have*

$$\begin{aligned} m &= \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) \leq 0\} \\ &= \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\}. \end{aligned} \quad (2.2)$$

*Proof.* For any  $\varphi \in H^1$ ,  $\varphi \neq 0$  with  $K(\varphi) = 0$ , we have  $E(\varphi) = H(\varphi)$ , this implies that

$$\begin{aligned} m &= \inf\{E(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) = 0\} \\ &\geq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) \leq 0\}. \end{aligned} \quad (2.3)$$

On the other hand, for any  $\varphi \in H^1$ ,  $\varphi \neq 0$  with  $K(\varphi) < 0$ , by Lemma 2.6, Lemma 2.7 and the continuity of  $K$  in  $\lambda$ , we know that there exists a  $\lambda_0 < 0$  such that

$$K(\varphi_{3,-2}^{\lambda_0}) = 0,$$

then by  $\mathcal{L}H \geq 0$ , we have

$$E(\varphi_{3,-2}^{\lambda_0}) = H(\varphi_{3,-2}^{\lambda_0}) \leq H(\varphi_{3,-2}^0) = H(\varphi).$$

Therefore,

$$\begin{aligned} & \inf\{E(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) = 0\} \\ & \leq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\}. \end{aligned} \quad (2.4)$$

By (2.3) and (2.4), we have

$$\begin{aligned} & \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) \leq 0\} \\ & \leq m \leq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\}. \end{aligned}$$

In order to show (2.2), it suffices to show that

$$\begin{aligned} & \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) \leq 0\} \\ & \geq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\}. \end{aligned} \quad (2.5)$$

For any  $\varphi \in H^1$ ,  $\varphi \neq 0$  with  $K(\varphi) \leq 0$ . By Lemma 2.8, we know that

$$\mathcal{L}K(\varphi) = \bar{\mu}K(\varphi) - \int_{\mathbb{R}^3} \left(4|\nabla\varphi|^2 + 12|\varphi|^6\right) dx < 0,$$

then for any  $\lambda > 0$  we have

$$K(\varphi_{3,-2}^\lambda) < 0,$$

and as  $\lambda \rightarrow 0$

$$H(\varphi_{3,-2}^\lambda) = \int_{\mathbb{R}^3} \left( \frac{e^{4\lambda}}{6} |\nabla\varphi|^2 + \frac{e^{12\lambda}}{6} |\varphi|^6 \right) dx \longrightarrow H(\varphi).$$

This shows (2.5), and completes the proof.  $\square$

Next we will use the ( $\dot{H}^1$ -invariant) scaling argument to remove the  $L^4$  term (the lower regularity quantity than  $\dot{H}^1$ ) in  $K$ , that is, to replace the constrained condition  $K(\varphi) < 0$  with  $K^c(\varphi) < 0$ , where

$$K^c(\varphi) := \int_{\mathbb{R}^3} (2|\nabla\varphi|^2 - 2|\varphi|^6) dx.$$

In fact, we have

**Lemma 2.10.** *For the minimization  $m$  in (1.5), we have*

$$\begin{aligned} m &= \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) < 0\} \\ &= \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) \leq 0\}. \end{aligned}$$

*Proof.* Since  $K^c(\varphi) \leq K(\varphi)$ , it is obvious that

$$m = \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\}$$

$$\geq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) < 0\}.$$

Hence in order to show the first equality, it suffices to show that

$$\begin{aligned} & \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) < 0\} \\ & \leq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) < 0\}. \end{aligned} \quad (2.6)$$

To do so, for any  $\varphi \in H^1$ ,  $\varphi \neq 0$  with  $K^c(\varphi) < 0$ , taking

$$\varphi_{1,-2}^\lambda(x) = e^\lambda \varphi(e^{2\lambda}x),$$

we have  $\varphi_{1,-2}^\lambda \in H^1$  and  $\varphi_{1,-2}^\lambda \neq 0$  for any  $\lambda > 0$ . In addition, we have

$$\begin{aligned} K(\varphi_{1,-2}^\lambda) &= \int_{\mathbb{R}^3} \left( 2|\nabla \varphi|^2 - 2|\varphi|^6 + \frac{3}{2}e^{-2\lambda}|\varphi|^4 \right) dx \longrightarrow K^c(\varphi), \\ H(\varphi_{1,-2}^\lambda) &= \int_{\mathbb{R}^3} \left( \frac{1}{6}|\nabla \varphi|^2 + \frac{1}{6}|\varphi|^6 \right) dx = H(\varphi), \end{aligned}$$

as  $\lambda \rightarrow +\infty$ . This gives (2.6), and completes the proof of the first equality.

For the second equality, it is obvious that

$$\begin{aligned} & \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) < 0\} \\ & \geq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) \leq 0\}, \end{aligned}$$

hence we only need to show that

$$\begin{aligned} & \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) < 0\} \\ & \leq \inf\{H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K^c(\varphi) \leq 0\}. \end{aligned} \quad (2.7)$$

To do this, we use the ( $L^2$ -invariant) scaling argument. For any  $\varphi \in H^1$ ,  $\varphi \neq 0$  with  $K^c(\varphi) \leq 0$ , we have  $\varphi_{3,-2}^\lambda \in H^1$ ,  $\varphi_{3,-2}^\lambda \neq 0$ . In addition, by

$$\begin{aligned} \mathcal{L}K^c(\varphi) &= \int_{\mathbb{R}^3} \left( 8|\nabla \varphi|^2 - 24|\varphi|^6 \right) dx = 4K^c(\varphi) - 16\|\varphi\|_{L^6}^6 < 0, \\ H(\varphi_{3,-2}^\lambda) &= \int_{\mathbb{R}^3} \left( \frac{e^{4\lambda}}{6}|\nabla \varphi|^2 + \frac{e^{12\lambda}}{6}|\varphi|^6 \right) dx, \end{aligned}$$

we have  $K^c(\varphi_{3,-2}^\lambda) < 0$  for any  $\lambda > 0$ , and

$$H(\varphi_{3,-2}^\lambda) \rightarrow H(\varphi), \text{ as } \lambda \rightarrow 0.$$

This implies (2.7) and completes the proof.  $\square$

After these preparations, we can now make use of the sharp Sobolev constant in [1, 35] to compute the minimization  $m$  of (1.5), which also shows Proposition 1.1.

**Lemma 2.11.** *For the minimization  $m$  in (1.5), we have*

$$m = E^c(W).$$

*Proof.* By Lemma 2.10, we have

$$\begin{aligned} m &= \inf \left\{ \frac{1}{6} \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + |\varphi|^6) dx \mid \varphi \in H^1, \varphi \neq 0, \|\nabla \varphi\|_{L^2}^2 \leq \|\varphi\|_{L^6}^6 \right\} \\ &\geq \inf \left\{ \int_{\mathbb{R}^3} \frac{1}{6} (|\nabla \varphi|^2 + |\varphi|^6) + \frac{1}{6} (|\nabla \varphi|^2 - |\varphi|^6) dx \mid \varphi \in H^1, \varphi \neq 0, \|\nabla \varphi\|_{L^2}^2 \leq \|\varphi\|_{L^6}^6 \right\} \end{aligned}$$

where the equality holds if and only if the minimization is taken by some  $\varphi$  with  $\|\nabla \varphi\|_{L^2}^2 = \|\varphi\|_{L^6}^6$ . While

$$\begin{aligned} &\inf \left\{ \int_{\mathbb{R}^3} \frac{1}{3} |\nabla \varphi|^2 dx \mid \varphi \in H^1, \varphi \neq 0, \|\nabla \varphi\|_{L^2}^2 \leq \|\varphi\|_{L^6}^6 \right\} \\ &= \inf \left\{ \frac{1}{3} \|\nabla \varphi\|_{L^2}^2 \left( \frac{\|\nabla \varphi\|_{L^2}^2}{\|\varphi\|_{L^6}^6} \right)^{1/2} \mid \varphi \in H^1, \varphi \neq 0 \right\} \\ &= \inf \left\{ \frac{1}{3} \left( \frac{\|\nabla \varphi\|_{L^2}}{\|\varphi\|_{L^6}} \right)^3 \mid \varphi \in H^1, \varphi \neq 0 \right\} \\ &= \inf \left\{ \frac{1}{3} \left( \frac{\|\nabla \varphi\|_{L^2}}{\|\varphi\|_{L^6}} \right)^3 \mid \varphi \in \dot{H}^1, \varphi \neq 0 \right\} = \frac{1}{3} (C_3^*)^{-3}. \end{aligned}$$

where we use the density property  $H^1 \hookrightarrow \dot{H}^1$  in the last second equality and that  $C_3^*$  is the sharp Sobolev constant in  $\mathbb{R}^3$ , that is,

$$\|\varphi\|_{L_x^6} \leq C_3^* \|\nabla \varphi\|_{L_x^2}, \quad \forall \varphi \in \dot{H}^1(\mathbb{R}^3),$$

and the equality can be attained by the ground state  $W$  of the following elliptic equation

$$-\Delta W = |W|^4 W.$$

This implies that  $\frac{1}{3} (C_3^*)^{-3} = E^c(W)$ . The proof is completed.  $\square$

After the computation of the minimization  $m$  in (1.5), we next give some variational estimates.

**Lemma 2.12.** *For any  $\varphi \in H^1$  with  $K(\varphi) \geq 0$ , we have*

$$\int_{\mathbb{R}^3} \left( \frac{1}{6} |\nabla \varphi|^2 + \frac{1}{6} |\varphi|^6 \right) dx \leq E(\varphi) \leq \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{4} |\varphi|^4 \right) dx. \quad (2.8)$$

*Proof.* On one hand, the right hand side of (2.8) is trivial. On the other hand, by the definition of  $E$  and  $K$ , we have

$$E(\varphi) = \int_{\mathbb{R}^3} \left( \frac{1}{6} |\nabla \varphi|^2 + \frac{1}{6} |\varphi|^6 \right) dx + \frac{1}{6} K(\varphi),$$

which implies the left hand side of (2.8).  $\square$

At the last of this section, we give the uniform bounds on the scaling derivative functional  $K(\varphi)$  with the energy  $E(\varphi)$  below the threshold  $m$ , which plays an important role for the blow-up and scattering analysis in Section 3 and Section 6.

**Lemma 2.13.** *For any  $\varphi \in H^1$  with  $E(\varphi) < m$ .*

(1) *If  $K(\varphi) < 0$ , then*

$$K(\varphi) \leq -6(m - E(\varphi)). \quad (2.9)$$

(2) *If  $K(\varphi) \geq 0$ , then*

$$K(\varphi) \geq \min \left( 6(m - E(\varphi)), \frac{2}{3} \|\nabla \varphi\|_{L^2}^2 + \frac{1}{2} \|\varphi\|_{L^4}^4 \right). \quad (2.10)$$

*Proof.* By Lemma 2.8, for any  $\varphi \in H^1$ , we have

$$\mathcal{L}^2 E(\varphi) = \bar{\mu} \mathcal{L} E(\varphi) - 4 \|\nabla \varphi\|_{L^2}^2 - 12 \|\varphi\|_{L^6}^6.$$

Let  $j(\lambda) = E(\varphi_{3,-2}^\lambda)$ , then we have

$$j''(\lambda) = \bar{\mu} j'(\lambda) - 4e^{4\lambda} \|\nabla \varphi\|_{L^2}^2 - 12e^{12\lambda} \|\varphi\|_{L^6}^6. \quad (2.11)$$

**Case I:** If  $K(\varphi) < 0$ , then by (2.1), Lemma 2.7 and the continuity of  $K$  in  $\lambda$ , there exists a negative number  $\lambda_0 < 0$  such that  $K(\varphi_{3,-2}^{\lambda_0}) = 0$ , and

$$K(\varphi_{3,-2}^\lambda) < 0, \quad \forall \lambda \in (\lambda_0, 0).$$

By (1.5), we obtain  $j(\lambda_0) = E(\varphi_{3,-2}^{\lambda_0}) \geq m$ . Now by integrating (2.11) over  $[\lambda_0, 0]$ , we have

$$\int_{\lambda_0}^0 j''(\lambda) d\lambda \leq \bar{\mu} \int_{\lambda_0}^0 j'(\lambda) d\lambda,$$

which implies that

$$K(\varphi) = j'(0) - j'(\lambda_0) \leq \bar{\mu} (j(0) - j(\lambda_0)) \leq -\bar{\mu} (m - E(\varphi)),$$

which implies (2.9).

**Case II:**  $K(\varphi) \geq 0$ . We divide it into two subcases:

When  $2\bar{\mu}K(\varphi) \geq 12\|\varphi\|_{L^6}^6$ . Since

$$12 \int_{\mathbb{R}^3} |\varphi|^6 dx = -6K(\varphi) + \int_{\mathbb{R}^3} \left(12|\nabla\varphi|^2 + 9|\varphi|^4\right) dx,$$

then we have

$$2\bar{\mu}K(\varphi) \geq -6K(\varphi) + \int_{\mathbb{R}^3} \left(12|\nabla\varphi|^2 + 9|\varphi|^4\right) dx,$$

which implies that

$$K(\varphi) \geq \frac{2}{3}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^4}^4.$$

When  $2\bar{\mu}K(\varphi) \leq 12\|\varphi\|_{L^6}^6$ . By (2.11), we have for  $\lambda = 0$

$$\begin{aligned} 0 &< 2\bar{\mu}j'(\lambda) < 12e^{12\lambda}\|\varphi\|_{L^6}^6, \\ j''(\lambda) &= \bar{\mu}j'(\lambda) - 4e^{4\lambda}\|\nabla\varphi\|_{L^2}^2 - 12e^{12\lambda}\|\varphi\|_{L^6}^6 \leq -\bar{\mu}j'(\lambda). \end{aligned} \quad (2.12)$$

By the continuity of  $j'$  and  $j''$  in  $\lambda$ , we know that  $j'$  is an accelerating decreasing function as  $\lambda$  increases until  $j'(\lambda_0) = 0$  for some finite number  $\lambda_0 > 0$  and (2.12) holds on  $[0, \lambda_0]$ .

By  $K(\varphi_{3,-2}^{\lambda_0}) = j'(\lambda_0) = 0$ , we know that

$$E(\varphi_{3,-2}^{\lambda_0}) \geq m.$$

Now integrating (2.12) over  $[0, \lambda_0]$ , we obtain that

$$-K(\varphi) = j'(\lambda_0) - j'(0) \leq -\bar{\mu}(j(\lambda_0) - j(0)) \leq -\bar{\mu}(m - E(\varphi)).$$

This completes the proof.  $\square$

### 3. PART I: BLOW UP FOR $\mathcal{K}^-$

In this section, we prove the blow-up result of Theorem 1.3. We can also refer to [33]. Now let  $\phi$  be a smooth, radial function satisfying  $\partial_r^2\phi(r) \leq 2$ ,  $\phi(r) = r^2$  for  $r \leq 1$ , and  $\phi(r)$  is constant for  $r \geq 3$ . For some  $R$ , we define

$$V_R(t) := \int_{\mathbb{R}^3} \phi_R(x)|u(t, x)|^2 dx, \quad \phi_R(x) = R^2\phi\left(\frac{|x|}{R}\right).$$

By Lemma 2.5,  $\Delta\phi_R(r) = 6$  for  $r \leq R$ , and  $\Delta^2\phi_R(r) = 0$  for  $r \leq R$ , we have

$$\begin{aligned} \partial_t^2 V_R(t) &= 4 \int_{\mathbb{R}^3} \phi_R''(r)|\nabla u(t, x)|^2 dx - \int_{\mathbb{R}^3} (\Delta^2\phi_R)(x)|u(t, x)|^2 dx \\ &\quad - \frac{4}{3} \int_{\mathbb{R}^3} (\Delta\phi_R)|u(t, x)|^6 dx + \int_{\mathbb{R}^3} (\Delta\phi_R)|u(t, x)|^4 dx \end{aligned}$$

$$\begin{aligned} &\leq 4 \int_{\mathbb{R}^3} \left( 2|\nabla u(t)|^2 - 2|u(t)|^6 + \frac{3}{2}|u(t)|^4 \right) dx \\ &\quad + \frac{c}{R^2} \int_{R \leq |x| \leq 3R} |u(t)|^2 dx + c \int_{R \leq |x| \leq 3R} \left( |u(t)|^4 + |u(t)|^6 \right) dx. \end{aligned}$$

By the Gagliardo-Nirenberg and radial Sobolev inequalities, we have

$$\begin{aligned} \|f\|_{L^4(|x| \geq R)}^4 &\leq \frac{c}{R^2} \|f\|_{L^2(|x| \geq R)}^3 \|\nabla f\|_{L^2(|x| \geq R)}, \\ \|f\|_{L^\infty(|x| \geq R)} &\leq \frac{c}{R} \|f\|_{L^2(|x| \geq R)}^{1/2} \|\nabla f\|_{L^2(|x| \geq R)}^{1/2}. \end{aligned}$$

Therefore, by mass conservation and Young's inequality, we know that for any  $\epsilon > 0$  there exist sufficiently large  $R$  such that

$$\begin{aligned} \partial_t^2 V_R(t) &\leq 4K(u(t)) + \epsilon \|\nabla u(t, x)\|_{L^2}^2 + \epsilon^2 \\ &= 48E(u) - (16 - \epsilon) \|\nabla u(t)\|_{L^2}^2 - 6\|u(t)\|_{L^4}^4 + \epsilon^2 \end{aligned} \quad (3.1)$$

By  $K(u) < 0$ , mass and energy conservations, Lemma 2.13 and the continuity argument, we know that for any  $t \in I$ , we have

$$K(u(t)) \leq -6(m - E(u(t))) < 0.$$

By Lemma 2.9, we have

$$m \leq H(u(t)) < \frac{1}{3} \|u(t)\|_{L^6}^6.$$

where we have used the fact that  $K(u(t)) < 0$  in the second inequality. By the fact  $m = \frac{1}{3} (C_3^*)^{-3}$  and the Sharp Sobolev inequality, we have

$$\|\nabla u(t)\|_{L^2}^6 \geq (C_3^*)^{-6} \|u(t)\|_{L^6}^6 > (C_3^*)^{-9},$$

which implies that  $\|\nabla u(t)\|_{L^2}^2 > 3m$ .

In addition, by  $E(u_0) < m$  and energy conservation, there exists  $\delta_1 > 0$  such that  $E(u(t)) \leq (1 - \delta_1)m$ . Thus, if we choose  $\epsilon$  sufficiently small, we have

$$\partial_t^2 V_R(t) \leq 48(1 - \delta_1)m - 3(16 - \epsilon)m + \epsilon^2 \leq -24\delta_1 m,$$

which implies that  $u$  must blow up at finite time.  $\square$

#### 4. PERTURBATION THEORY

In this part, we give the perturbation theory of the solution of (1.1) with the global space-time estimate. First we denote the space-time space  $ST(I)$  on the time interval



$I$  by

$$ST(I) := \left( L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9 \cap L_t^6 \dot{B}_{18/7,2}^{1/2} \cap L_{t,x}^5 \right) (I \times \mathbb{R}^3),$$

$$ST^*(I) := \left( L_t^2 \dot{B}_{18/11,2}^{1/3} \cap L_t^2 \dot{B}_{6/5,2}^{1/2} \right) (I \times \mathbb{R}^3).$$

The main result in this section is the following.

**Proposition 4.1.** *Let  $I$  be a compact time interval and let  $w$  be an approximate solution to (1.1) on  $I \times \mathbb{R}^3$  in the sense that*

$$i\partial_t w + \Delta w = -|w|^4 w + |w|^2 w + e$$

for some suitable small function  $e$ . Assume that for some constants  $L, E_0 > 0$ , we have

$$\|w\|_{ST(I)} \leq L, \quad \|w(t_0)\|_{H_x^1(\mathbb{R}^3)} \leq E_0$$

for some  $t_0 \in I$ . Let  $u(t_0)$  close to  $w(t_0)$  in the sense that for some  $E' > 0$ , we have

$$\|u(t_0) - w(t_0)\|_{H_x^1} \leq E'.$$

Assume also that for some  $\varepsilon$ , we have

$$\|e^{i(t-t_0)\Delta}(u(t_0) - w(t_0))\|_{ST(I)} \leq \varepsilon, \quad \|e\|_{ST^*(I)} \leq \varepsilon, \quad (4.1)$$

where  $0 < \varepsilon \leq \varepsilon_0 = \varepsilon_0(E_0, E', L)$  is a small constant. Then there exists a solution  $u$  to (1.1) on  $I \times \mathbb{R}^3$  with initial data  $u(t_0)$  at time  $t = t_0$  satisfying

$$\|u - w\|_{ST(I)} \leq C(E_0, E', L) \varepsilon, \quad \text{and} \quad \|u\|_{ST(I)} \leq C(E_0, E', L).$$

*Proof.* Since  $w \in ST(I)$ , there exists a partition of the right half of  $I$  at  $t_0$ :

$$t_0 < t_1 < \cdots < t_N, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_N),$$

such that  $N \leq C(L, \delta)$  and for any  $j = 0, 1, \dots, N-1$ , we have

$$\|w\|_{ST(I_j)} \leq \delta \ll 1. \quad (4.2)$$

The estimate on the left half of  $I$  at  $t_0$  is analogue, we omit it.

Let

$$\gamma(t, x) = u(t, x) - w(t, x),$$

$$\gamma_j(t, x) = e^{i(t-t_j)\Delta} \left( u(t_j, x) - w(t_j, x) \right),$$

then  $\gamma$  satisfies the following difference equation

$$i\gamma_t + \Delta\gamma = O(w^4\gamma + w^3\gamma^2 + w^2\gamma^3 + w\gamma^4 + \gamma^5 + w^2\gamma + w\gamma^2 + \gamma^3) - e,$$

which implies that

$$\begin{aligned}\gamma(t) &= \gamma_j(t) - i \int_{t_j}^t e^{i(t-s)\Delta} \left( O(w^4\gamma + w^3\gamma^2 + w^2\gamma^3 + w\gamma^4 + \gamma^5 + w^2\gamma + w\gamma^2 + \gamma^3) - e \right) ds, \\ \gamma_{j+1}(t) &= \gamma_j(t) - i \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} \left( O(w^4\gamma + w^3\gamma^2 + w^2\gamma^3 + w\gamma^4 + \gamma^5 + w^2\gamma + w\gamma^2 + \gamma^3) - e \right) ds.\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}& \|\gamma - \gamma_j\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2}) \cap L_{t,x}^5(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{L_t^6(\mathbb{R}; \dot{B}_{18/7,2}^{1/2}) \cap L_{t,x}^5(\mathbb{R} \times \mathbb{R}^3)} \quad (4.3) \\& \lesssim \|O(w^4\gamma + w^3\gamma^2 + w^2\gamma^3 + w\gamma^4 + \gamma^5)\|_{L_t^2(I_j; \dot{B}_{6/5,2}^{1/2})} \\& \quad + \|w^2\gamma + w\gamma^2 + \gamma^3\|_{L_t^2(I_j; \dot{B}_{6/5,2}^{1/2})} + \|e\|_{L_t^2(I_j; \dot{B}_{6/5,2}^{1/2})} \\& \lesssim \|w\|_{L_t^{12}(I_j; L_x^9)}^4 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L_t^{12}(I_j; L_x^9)}^3 \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L_t^{12}(I_j; L_x^9)} \\& \quad + \|w\|_{L_t^{12}(I_j; L_x^9)}^3 \|\gamma\|_{L_t^{12}(I_j; L_x^9)} \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L_t^{12}(I_j; L_x^9)}^2 \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^2 \\& \quad + \|w\|_{L_t^{12}(I_j; L_x^9)}^2 \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^2 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L_t^{12}(I_j; L_x^9)} \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^3 \\& \quad + \|w\|_{L_t^{12}(I_j; L_x^9)} \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^3 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^4 \\& \quad + \|\gamma\|_{L_t^{12}(I_j; L_x^9)}^4 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \\& \quad + \|w\|_{L^6(I_j; L_x^{9/2})}^2 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L^6(I_j; L_x^{9/2})} \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L^6(I_j; L_x^{9/2})} \\& \quad + \|w\|_{L^6(I_j; L_x^{9/2})} \|\gamma\|_{L^6(I_j; L_x^{9/2})} \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|w\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} \|\gamma\|_{L^6(I_j; L_x^{9/2})}^2 \\& \quad + \|\gamma\|_{L^6(I_j; L_x^{9/2})}^2 \|\gamma\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})} + \|e\|_{L_t^2(I_j; \dot{B}_{6/5,2}^{1/2})}.\end{aligned}$$

At the same time, by Lemma 2.3, we have

$$\begin{aligned}& \|\gamma - \gamma_j\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3}) \cap L_t^{12}(I_j; L_x^9)} + \|\gamma_{j+1} - \gamma_j\|_{L_t^{10}(\mathbb{R}; \dot{B}_{90/19,2}^{1/3}) \cap L_t^{12}(\mathbb{R}; L_x^9)} \quad (4.4) \\& \lesssim \|O(w^4\gamma + w^3\gamma^2 + w^2\gamma^3 + w\gamma^4 + \gamma^5 + w^2\gamma + w\gamma^2 + \gamma^3)\|_{L^2(I_j; \dot{B}_{18/7,2}^{1/3})} + \|e\|_{L^2(I_j; \dot{B}_{18/7,2}^{1/3})} \\& \lesssim \|w\|_{L_{t,x}^{10}(I_j)}^4 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_{t,x}^{10}(I_j)}^3 \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^{10}(I_j)} \\& \quad + \|w\|_{L_{t,x}^{10}(I_j)}^3 \|\gamma\|_{L_{t,x}^{10}(I_j)} \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_{t,x}^{10}(I_j)}^2 \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^{10}(I_j)}^2 \\& \quad + \|w\|_{L_{t,x}^{10}(I_j)}^2 \|\gamma\|_{L_{t,x}^{10}(I_j)}^2 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_{t,x}^{10}(I_j)} \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^{10}(I_j)}^3 \\& \quad + \|w\|_{L_{t,x}^{10}(I_j)} \|\gamma\|_{L_{t,x}^{10}(I_j)}^3 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^{10}(I_j)}^4\end{aligned}$$

$$\begin{aligned}
& + \|\gamma\|_{L_{t,x}^{10}(I_j)}^4 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \\
& + \|w\|_{L_{t,x}^5(I_j)}^2 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_{t,x}^5(I_j)} \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^5(I_j)} \\
& + \|w\|_{L_{t,x}^5(I_j)} \|\gamma\|_{L_{t,x}^5(I_j)} \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|w\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} \|\gamma\|_{L_{t,x}^5(I_j)}^2 \\
& + \|\gamma\|_{L_{t,x}^5(I_j)}^2 \|\gamma\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})} + \|e\|_{L^2(I_j; \dot{B}_{18/11,2}^{1/3})}.
\end{aligned}$$

By the interpolation, we have

$$\|f\|_{L^6(I_j; L_x^{9/2})} \lesssim \|f\|_{L_t^6(I_j; \dot{B}_{18/7,2}^{1/2})}, \quad \|f\|_{L_{t,x}^{10}(I_j)} \lesssim \|f\|_{L_t^{10}(I_j; \dot{B}_{90/19,2}^{1/3})}.$$

Therefore, assuming that

$$\|\gamma\|_{ST(I_j)} \leq \delta \ll 1, \quad \forall j = 0, 1, \dots, N-1, \quad (4.5)$$

then by (4.2), (4.3) and (4.4), we have

$$\|\gamma\|_{ST(I_j)} + \|\gamma_{j+1}\|_{ST(t_{j+1}, t_N)} \leq C \|\gamma_j\|_{ST(t_j, t_N)} + \varepsilon,$$

for some absolute constant  $C > 0$ . By (4.1) and iteration on  $j$ , we get

$$\|\gamma\|_{ST(I)} \leq (2C)^N \varepsilon \leq \frac{\delta}{2},$$

if we choose  $\varepsilon_0$  sufficiently small. Hence the assumption (4.5) is justified by continuity in  $t$  and induction on  $j$ . then repeating the estimate (4.3) and (4.4) once again, we can obtain the  $ST$ -norm estimate on  $\gamma$ , which implies the Strichartz estimate on  $u$ .  $\square$

## 5. PROFILE DECOMPOSITION

In this part, we will use the method in [2, 17, 21] to show the linear and nonlinear profile decompositions of the sequences of radial,  $H^1$ -bounded solutions of (1.1), which will be used to construct the critical element (minimal energy non-scattering solution) and show its properties, especially the compactness. In order to do it, we now introduce the complex-valued function  $\vec{v}(t, x)$  by

$$\vec{v}(t, x) = \langle \nabla \rangle v(t, x), \quad v(t, x) = \langle \nabla \rangle^{-1} \vec{v}(t, x).$$

Given  $(t_n^j, h_n^j) \in \mathbb{R} \times (0, 1]$ , let  $\tau_n^j$ ,  $T_n^j$  denote the scaled time drift, the scaling transformation, defined by

$$\tau_n^j = -\frac{t_n^j}{(h_n^j)^2}, \quad T_n^j \varphi(x) = \frac{1}{(h_n^j)^{3/2}} \varphi\left(\frac{x}{h_n^j}\right).$$

We also introduce the set of Fourier multipliers on  $\mathbb{R}^3$ .

$$\mathcal{MC} = \{\mu = \mathcal{F}^{-1}\tilde{\mu}\mathcal{F} \mid \tilde{\mu} \in C(\mathbb{R}^3), \exists \lim_{|\xi| \rightarrow +\infty} \tilde{\mu}(\xi) \in \mathbb{R}\}.$$

**5.1. Linear profile decomposition.** In this subsection, we show the profile decomposition with the scaling parameter of a sequence of the radial, free Schrödinger solutions in the energy space  $H^1(\mathbb{R}^3)$ , which implies the profile decomposition of a sequence of radial initial data.

**Proposition 5.1.** *Let*

$$\vec{v}_n(t, x) = e^{it\Delta} \vec{v}_n(0)$$

*be a sequence of the radial solutions of the free Schrödinger equation with bounded  $L^2$  norm. Then up to a subsequence, there exist  $K \in \{0, 1, 2, \dots, \infty\}$ , radial functions  $\{\varphi^j\}_{j \in [0, K)} \subset L^2(\mathbb{R}^3)$  and  $\{t_n^j, h_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R} \times (0, 1]$  satisfying*

$$\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x), \quad (5.1)$$

*where  $\vec{v}_n^j(t, x) = e^{i(t-t_n^j)\Delta} T_n^j \varphi^j$ , and*

$$\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \|\vec{w}_n^k\|_{L_t^\infty(\mathbb{R}; B_{\infty, \infty}^{-3/2}(\mathbb{R}^3))} = 0, \quad (5.2)$$

*and for any Fourier multiplier  $\mu \in \mathcal{MC}$ , any  $l < j < k \leq K$  and any  $t \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow +\infty} \left( \log \left| \frac{h_n^j}{h_n^l} \right| + \left| \frac{t_n^j - t_n^l}{(h_n^l)^2} \right| \right) = \infty, \quad (5.3)$$

$$\lim_{n \rightarrow +\infty} \langle \mu \vec{v}_n^l(t), \mu \vec{v}_n^j(t) \rangle_{L_x^2} = \lim_{n \rightarrow +\infty} \langle \mu \vec{v}_n^j(t), \mu \vec{w}_n^k(t) \rangle_{L_x^2} = 0. \quad (5.4)$$

*Moreover, each sequence  $\{h_n^j\}_{n \in \mathbb{N}}$  is either going to 0 or identically 1 for all  $n$ .*

**Remark 5.2.** We call  $\vec{v}_n^j$  and  $\vec{w}_n^k$  the free concentrating wave and the remainder, respectively. From (5.4), we have the following asymptotic orthogonality

$$\lim_{n \rightarrow +\infty} \left( \|\mu \vec{v}_n(t)\|_{L^2}^2 - \sum_{j=0}^{k-1} \|\mu \vec{v}_n^j(t)\|_{L^2}^2 - \|\mu \vec{w}_n^k(t)\|_{L^2}^2 \right) = 0. \quad (5.5)$$

*Proof of Proposition 5.1.* Let

$$\nu := \overline{\lim}_{n \rightarrow \infty} \|\vec{v}_n\|_{L_t^\infty B_{\infty, \infty}^{-3/2}} = \overline{\lim}_{n \rightarrow \infty} \sup_{\substack{(t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ k \geq 0}} 2^{-3k/2} |\Lambda_k * \vec{v}_n(t, x)|.$$

If  $\nu = 0$ , then we have done with  $K = 0$ .

Otherwise,  $\nu = \overline{\lim}_{n \rightarrow \infty} \|\vec{v}_n\|_{L_t^\infty B_{\infty, \infty}^{-3/2}} > 0$ . By the radial Gagliardo-Nirenberg inequality and the Bernstein inequality, we have

$$\begin{aligned} \sup_{\substack{t \in \mathbb{R}, |2^k x| \geq R, \\ k \geq 0}} 2^{-3k/2} |\Lambda_k * \vec{v}_n(t, x)| &\lesssim \sup_{k \geq 0} \frac{2^k 2^{-3k/2}}{R} \|\Lambda_k * \vec{v}_n(t, x)\|_{L_t^\infty L_x^2}^{1/2} \cdot \|\nabla \Lambda_k * \vec{v}_n(t, x)\|_{L_t^\infty L_x^2}^{1/2} \\ &\lesssim \sup_{k \geq 0} \frac{1}{R} \|\vec{v}_n(t, x)\|_{L_t^\infty L_x^2} \lesssim \frac{1}{R}. \end{aligned}$$

If taking  $R$  sufficiently large, we have

$$\sup_{t \in \mathbb{R}, |2^k x| \geq R, k \geq 0} 2^{-3k/2} |\Lambda_k * \vec{v}_n(t, x)| \leq \frac{1}{2} \nu.$$

thus, there exists a sequence  $(t_n, x_n, k_n)$  with  $k_n \geq 0$  and  $|2^{k_n} x_n| \leq R$  such that for large  $n$ ,

$$\frac{1}{2} \overline{\lim}_{n \rightarrow \infty} \|\vec{v}_n\|_{L_t^\infty B_{\infty, \infty}^{-3/2}} = \frac{1}{2} \nu \leq 2^{-3k_n/2} |\Lambda_{k_n} * \vec{v}_n(t_n, x_n)|.$$

Now we define  $h_n$  and  $\psi_n$  by  $h_n = 2^{-k_n} \in (0, 1]$  and

$$\begin{aligned} \vec{v}_n(t_n, x) &= (T_n \psi_n)(x - x_n) = \frac{1}{(h_n)^{3/2}} \psi_n\left(\frac{x - x_n}{h_n}\right) \\ &= T_n\left(\psi_n\left(x - \frac{x_n}{h_n}\right)\right). \end{aligned} \tag{5.6}$$

Since  $\|\psi_n\|_{L^2} = \|T_n \psi_n\|_{L^2} = \|\vec{v}_n(t_n)\|_{L^2} \leq C$ , then there exists some  $\psi \in L^2$ , such that, up to a subsequence, we have as  $n \rightarrow +\infty$

$$\frac{x_n}{h_n} \rightarrow x^0, \quad \text{and } \psi_n \rightharpoonup \psi \quad \text{weakly in } L^2. \tag{5.7}$$

On the other hand, if  $k_n = 0$ , we have

$$\begin{aligned} 2^{-3k_n/2} |\Lambda_{k_n} * \vec{v}_n(t_n, x_n)| &= \int_{\mathbb{R}^3} \Lambda_0(y) 2^{-3k_n/2} \vec{v}_n\left(t_n, x_n - \frac{y}{2^{k_n}}\right) dy \\ &= \int_{\mathbb{R}^3} \Lambda_0(y) \psi_n(-y) dy \\ &\longrightarrow \int_{\mathbb{R}^3} \Lambda_0(y) \psi(-y) dy \lesssim \|\psi\|_{L^2}. \end{aligned}$$

By the same way, if  $k_n \geq 1$ , we have

$$\begin{aligned} 2^{-3k_n/2} |\Lambda_{k_n} * \vec{v}_n(t_n, x_n)| &= \int_{\mathbb{R}^3} \Lambda_{(0)}(y) 2^{-3k_n/2} \vec{v}_n\left(t_n, x_n - \frac{y}{2^{k_n}}\right) dy \\ &= \int_{\mathbb{R}^3} \Lambda_{(0)}(y) \psi_n(-y) dy \end{aligned}$$

$$\longrightarrow \int_{\mathbb{R}^3} \Lambda_{(0)}(y) \psi(-y) dy \lesssim \|\psi\|_{L^2}.$$

If  $h_n \rightarrow 0$ , then we take

$$(t_n^0, h_n^0) = (t_n, h_n), \quad \varphi^0(x) = \psi(x - x^0),$$

otherwise, up to a subsequence, we may assume that  $h_n \rightarrow h_\infty$  for some  $h_\infty \in (0, 1]$ , and take

$$(t_n^0, h_n^0) = (t_n, 1), \quad \varphi^0(x) = \frac{1}{(h_\infty)^{3/2}} \psi\left(\frac{x}{h_\infty} - x^0\right),$$

then

$$T_n \left( \psi\left(x - \frac{x_n}{h_n}\right) \right) - T_n^0 \varphi^0(x) \longrightarrow 0 \quad \text{strongly in } L^2. \quad (5.8)$$

In addition, since  $\vec{v}_n(t_n, x) = (T_n \psi_n)(x - x_n)$  is radial, so is  $\varphi^0(x)$ .

Let  $\vec{v}_n^0(t, x) = e^{i(t-t_n^0)\Delta} T_n^0 \varphi^0$ , we define  $\vec{w}_n^1$  by

$$\vec{v}_n(t, x) = \vec{v}_n^0(t, x) + \vec{w}_n^1(t, x), \quad (5.9)$$

then by (5.7) and (5.8), we have

$$(T_n^0)^{-1} \vec{w}_n^1(t_n^0) = (T_n^0)^{-1} T_n \left( \psi_n\left(x - \frac{x_n}{h_n}\right) \right) - \varphi^0 \rightharpoonup 0 \quad \text{weakly in } L^2,$$

which implies that

$$\langle \mu \vec{v}_n^0(t), \mu \vec{w}_n^1(t) \rangle = \langle \mu \vec{v}_n^0(t_n^0), \mu \vec{w}_n^1(t_n^0) \rangle = \langle \mu_n^0 \varphi^0, \mu_n^0 (T_n^0)^{-1} \vec{w}_n^1(t_n^0) \rangle \longrightarrow 0,$$

where we used the conservation law in the first equality and the dominated convergence theorem and  $\mu_n^0(D) = \mu\left(\frac{D}{h_n^0}\right)$  in the last equality. It is the decomposition for  $k = 1$ .

Next we apply the above procedure to the sequence  $\vec{w}_n^1$  in place of  $\vec{v}_n$ , then either  $\overline{\lim}_{n \rightarrow \infty} \|\vec{w}_n^1\|_{L_t^\infty B_{\infty, \infty}^{-3/2}} = 0$  or we can find the next concentrating wave  $\vec{v}_n^1$  and the remainder  $\vec{w}_n^2$ , such that for some  $(t_n^1, h_n^1)$  with  $h_n^1 \in (0, 1]$  and radial function  $\varphi^1 \in L^2(\mathbb{R}^3)$ ,

$$\vec{w}_n^1(t, x) = \vec{v}_n^1(t, x) + \vec{w}_n^2(t, x) = e^{i(t-t_n^1)\Delta} T_n^1 \varphi^1(x) + \vec{w}_n^2(t, x), \quad (5.10)$$

and

$$\begin{aligned} \overline{\lim}_{n \rightarrow +\infty} \|\vec{w}_n^1\|_{L_t^\infty B_{\infty, \infty}^{-3/2}} &\lesssim \|\varphi^1\|_{L^2} = \|\vec{v}_n^1\|_{L^2}, \\ (T_n^1)^{-1} \vec{w}_n^2(t_n^1) &\rightharpoonup 0 \quad \text{weakly in } L^2 \implies \langle \mu \vec{v}_n^1(t), \mu \vec{w}_n^2(t) \rangle \longrightarrow 0. \end{aligned} \quad (5.11)$$

Iterating the above procedure, we can obtain the decomposition (5.1). It remains to show the properties (5.2), (5.3) and (5.4).

We first assume that (5.4) holds, then by (5.5) and the Cauchy criterion, we have

$$\lim_{n \rightarrow +\infty} \|\vec{w}_n^k\|_{L_t^\infty B_{\infty,\infty}^{-3/2}} \lesssim \|\varphi^k\|_{L^2} = \|\vec{v}_n^k\|_{L^2} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (5.12)$$

which implies (5.2).

Now we show (5.3) by contradiction. Suppose that (5.3) fails, then there exists a minimal  $(l, j)$  which violates (5.3). By extracting a subsequence, We may assume that  $h_n^l \rightarrow h_\infty^l$  and  $h_n^l/h_n^j$  and  $(t_n^l - t_n^j)/(h_n^l)^2$  all converge.

Now consider

$$\begin{aligned} (T_n^l)^{-1} \vec{w}_n^{l+1}(t_n^l) &= \sum_{m=l+1}^j (T_n^l)^{-1} \vec{v}_n^m(t_n^l) + (T_n^l)^{-1} \vec{w}_n^{j+1}(t_n^l) \\ &= \sum_{m=l+1}^j (T_n^l)^{-1} e^{i(t_n^l - t_n^m)\Delta} T_n^m \varphi^m + (T_n^l)^{-1} \vec{w}_n^{j+1}(t_n^l) \\ &= \sum_{m=l+1}^{j-1} S_n^{l,m} \varphi^m + S_n^{l,j} \varphi^j + (T_n^l)^{-1} \vec{w}_n^{j+1}(t_n^l), \end{aligned}$$

where

$$S_n^{l,m} = (T_n^l)^{-1} e^{i(t_n^l - t_n^m)\Delta} T_n^m = e^{i \frac{t_n^l - t_n^m}{(h_n^l)^2} \Delta} (T_n^l)^{-1} T_n^m := e^{it_n^{l,m} \Delta} T_n^{l,m}$$

with the sequence

$$t_n^{l,m} = \frac{t_n^l - t_n^m}{(h_n^l)^2}, \quad h_n^{l,m} = \frac{h_n^m}{h_n^l}. \quad (5.13)$$

By the procedure of constructing (5.1), as  $n \rightarrow +\infty$ , we have

$$\begin{aligned} (T_n^l)^{-1} \vec{w}_n^{l+1}(t_n^l) &\rightharpoonup 0 \quad \text{weakly in } L^2, \\ (T_n^j)^{-1} \vec{w}_n^{j+1}(t_n^j) &\rightharpoonup 0 \quad \text{weakly in } L^2, \end{aligned}$$

and by the asymptotic orthogonality (5.3) between  $m$  and  $l$  with  $m \in [l+1, j-1]$

$$S_n^{l,m} \varphi^m \rightharpoonup 0, \quad \forall m \in [l+1, j-1],$$

and by the convergence of  $h_n^l/h_n^j$  and  $(t_n^l - t_n^j)/(h_n^l)^2$ , we have  $S_n^{l,j} \varphi^j \rightarrow S_\infty^{l,j} \varphi^j$  and

$$(T_n^l)^{-1} \vec{w}_n^{j+1}(t_n^l) = S_n^{l,j} (T_n^j)^{-1} \vec{w}_n^{j+1}(t_n^j) \rightharpoonup 0 \quad \text{weakly in } L^2.$$

Then  $\varphi^j = 0$ , it is a contradiction. Thus we obtain the orthogonality (5.3).

Last we show (5.4). For  $j \neq l$ , we have

$$\langle \mu \vec{v}_n^l(t), \mu \vec{v}_n^j(t) \rangle_{L_x^2}$$

$$\begin{aligned}
&= \langle \mu \vec{v}_n^l(0), \mu \vec{v}_n^j(0) \rangle_{L_x^2} = \left\langle \mu e^{-it_n^l \Delta} T_n^l \varphi^l, \mu e^{-it_n^j \Delta} T_n^j \varphi^j \right\rangle_{L_x^2} \\
&= \left\langle e^{-it_n^l \Delta} T_n^l \mu_n^l \varphi^l, e^{-it_n^j \Delta} T_n^j \mu_n^j \varphi^j \right\rangle_{L_x^2} = \left\langle (T_n^j)^{-1} e^{i(t_n^j - t_n^l) \Delta} T_n^l \mu_n^l \varphi^l, \mu_n^j \varphi^j \right\rangle_{L_x^2} \\
&= \left\langle e^{i \frac{t_n^j - t_n^l}{(h_n^j)^2} \Delta} (T_n^j)^{-1} T_n^l \mu_n^l \varphi^l, \mu_n^j \varphi^j \right\rangle_{L_x^2} = \langle S_n^{j,l} \mu_n^l \varphi^l, \mu_n^j \varphi^j \rangle_{L_x^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty
\end{aligned}$$

where  $\tilde{\mu}_n^l(\xi) = \tilde{\mu}(\xi/h_n^l)$  and we used the fact that  $S_n^{j,l} \rightarrow 0$  weakly in  $L^2$  as  $n \rightarrow +\infty$  by (5.3). In addition, we have

$$\langle \mu \vec{v}_n^j(t), \mu \vec{w}_n^k(t) \rangle_{L_x^2} = \left\langle \mu \vec{v}_n^j(t), \mu \left( \vec{w}_n^{j+1}(t) - \sum_{m=j+1}^{k-1} \vec{v}_n^m(t) \right) \right\rangle_{L_x^2} \rightarrow 0$$

as  $n \rightarrow +\infty$ . This completes the proof of (5.4).  $\square$

After the orthogonality's proof of the linear energy, we begin with the orthogonal analysis for the nonlinear energy.

**Lemma 5.3.** *Let  $\vec{v}_n$  be a sequence of the radial solutions of the free Schrödinger equation. Let*

$$\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x)$$

*be the linear profile decomposition given by Proposition 5.1. Then we have*

$$\begin{aligned}
\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| M(v_n(0)) - \sum_{j=0}^{k-1} M(v_n^j(0)) - M(w_n^k(0)) \right| &= 0, \\
\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| E(v_n(0)) - \sum_{j=0}^{k-1} E(v_n^j(0)) - E(w_n^k(0)) \right| &= 0, \\
\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| K(v_n(0)) - \sum_{j=0}^{k-1} K(v_n^j(0)) - K(w_n^k(0)) \right| &= 0.
\end{aligned}$$

*Proof.* We can show that the quadratic terms in  $M$ ,  $E$  and  $K$  have the orthogonal decomposition by taking  $\mu = \frac{1}{\langle \nabla \rangle}$  and  $\mu = \frac{|\nabla|}{\langle \nabla \rangle}$  in Remark 5.2, thus it suffices to show that

$$\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| F_i(v_n(0)) - \sum_{j < k} F_i(v_n^j(0)) - F_i(w_n^k(0)) \right| = 0, \quad i = 1, 2,$$



where  $F_1$  and  $F_2$  are denoted by

$$F_1(u(t)) = \int_{\mathbb{R}^3} |u(t, x)|^6 dx, \quad F_2(u(t)) = \int_{\mathbb{R}^3} |u(t, x)|^4 dx.$$

In order to do so, we need re-arrange the linear concentrating wave with respect to its dispersive decay (whether  $\tau_n^j$  goes to  $\pm\infty$  or not for all  $j$ ). Let  $v_n^{<k}(0) = \sum_{j<k} v_n^j(0) =$

$$\begin{aligned} & \sum_{j<k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0) + \sum_{j<k, \tau_n^j \rightarrow \pm\infty} v_n^j(0) \text{ for some finite numbers } \tau_\infty^j \text{'s, then we have} \\ & \left| F_i(v_n(0)) - \sum_{j<k} F_i(v_n^j(0)) - F_i(w_n^k(0)) \right| \\ & \leq \left| F_i(v_n(0)) - F_i(v_n^{<k}(0)) \right| + \left| F_i(v_n^{<k}(0)) - F_i\left(\sum_{j<k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0)\right) \right| \\ & \quad + \left| F_i\left(\sum_{j<k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0)\right) - \sum_{j<k, \tau_n^j \rightarrow \tau_\infty^j} F_i(v_n^j(0)) \right| \\ & \quad + \left| \sum_{j<k, \tau_n^j \rightarrow \pm\infty} F_i(v_n^j(0)) \right| + |F_i(w_n^k(0))|. \end{aligned} \tag{5.14}$$

First, by (5.2) and interpolation, we have that

$$\lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \|w_n^k(0)\|_{L_x^p} = 0, \quad \forall \quad 2 < p \leq 6.$$

which implies that

$$\begin{aligned} & \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} |F_i(v_n(0)) - F_i(v_n^{<k}(0))| = 0, \\ & \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} |F_i(w_n^k(0))| = 0. \end{aligned}$$

Second by the dispersive estimate for  $v_n^j(0)$  with  $\tau_n^j \rightarrow \pm\infty$ , we have

$$\begin{aligned} & \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| F_i(v_n^{<k}(0)) - F_i\left(\sum_{j<k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0)\right) \right| = 0, \\ & \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| \sum_{j<k, \tau_n^j \rightarrow \pm\infty} F_i(v_n^j(0)) \right| = 0. \end{aligned}$$

Last we will use the approximation argument in [17] to show that every non-dispersive concentrating wave will get away from the others, which contributes to the orthogonality

of (5.14). Let  $\psi^j := e^{i\tau_\infty^j \Delta} \varphi^j \in L^2$ , we have

$$\left| F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0) \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(v_n^j(0)) \right| \quad (5.15)$$

$$\begin{aligned} &\leq \left| F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0) \right) - F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) \right| \\ &\quad + \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(v_n^j(0)) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \\ &\quad + \left| F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(\langle \nabla \rangle^{-1} T_n^j \psi^j) \right|. \end{aligned} \quad (5.16)$$

For those  $v_n^j(0)$  with  $\tau_n^j \rightarrow \tau_\infty^j$ , by the continuity of the operator  $e^{it\Delta}$  in  $t$  in  $H^1$ , we have

$$v_n^j(0) = \langle \nabla \rangle^{-1} e^{-it_n^j \Delta} T_n^j \varphi^j = \langle \nabla \rangle^{-1} T_n^j e^{i\tau_n^j \Delta} \varphi^j \longrightarrow \langle \nabla \rangle^{-1} T_n^j \psi^j \quad \text{in } H^1(\mathbb{R}^3),$$

which implies that

$$\begin{aligned} &\left| F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} v_n^j(0) \right) - F_i \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) \right| \rightarrow 0, \\ &\left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(v_n^j(0)) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_i(\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \rightarrow 0. \end{aligned}$$

Now we consider (5.16) for  $i = 1, 2$ , separately.

First for  $i = 2$ , we compute as following,

$$\begin{aligned} &\left| F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_2(\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \\ &\leq \left| F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j=1} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) \right| \\ &\quad + \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_2(\langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j=1} F_2(\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \end{aligned}$$

$$+ \left| F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} F_2 (\langle \nabla \rangle^{-1} T_n^j \psi^j) \right|.$$

For  $h_n^j \rightarrow 0$ , we have

$$\langle \nabla \rangle^{-1} T_n^j \psi^j \rightarrow 0 \quad \text{in } L^p, \quad \forall 2 \leq p < 6,$$

which implies that

$$\begin{aligned} & \left| F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) \right| \rightarrow 0, \\ & \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_2 (\langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} F_2 (\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \rightarrow 0. \end{aligned}$$

In addition, by the orthogonality (5.3), we know that there is at most one term  $\langle \nabla \rangle^{-1} T_n^j \psi^j$  with  $\tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1$ , hence

$$\left| F_2 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j, h_n^j = 1} F_2 (\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| = 0.$$

Now we consider the case  $i = 1$ , Let  $\widehat{\psi}^j = |\nabla|^{-1} \psi^j$  if  $h_n^j \rightarrow 0$ , and  $\widehat{\psi}^j = \langle \nabla \rangle^{-1} \psi^j$  if  $h_n^j \equiv 1$ , then we have  $\widehat{\psi}^j \in L_x^6$ , and

$$\begin{aligned} & \left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (\langle \nabla \rangle^{-1} T_n^j \psi^j) \right| \\ & \leq \left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) \right| \\ & \quad + \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (\langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widehat{\psi}^j) \right| \\ & \quad + \left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widehat{\psi}^j) \right|. \end{aligned}$$

Since

$$\| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j \widehat{\psi}^j \|_{L_x^6} = \begin{cases} \| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j |\nabla|^{-1} \psi^j \|_{L_x^6} & \text{if } h_n^j \rightarrow 0 \\ \| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j \langle \nabla \rangle^{-1} \psi^j \|_{L_x^6} & \text{if } h_n^j \equiv 1 \end{cases}$$

$$= \begin{cases} \|(\langle \nabla \rangle_n^j)^{-1} \psi^j - |\nabla|^{-1} \psi^j\|_{L_x^6} & \text{if } h_n^j \rightarrow 0 \\ 0 & \text{if } h_n^j \equiv 1 \end{cases} \\ \longrightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

which shows that

$$\left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} \langle \nabla \rangle^{-1} T_n^j \psi^j \right) - F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) \right| \longrightarrow 0, \\ \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (\langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widehat{\psi}^j) \right| \longrightarrow 0.$$

We further replace each  $\widehat{\psi}^j$  by the non-overlap terms  $\widetilde{\psi}_n^j$  with each other

$$\widetilde{\psi}_n^j = \widehat{\psi}^j \times \begin{cases} 0; & \exists l < j, \text{ such that } h_n^l < h_n^j \text{ and } \frac{x}{h_n^{j,l}} \in \text{supp } \widehat{\psi}^l, \\ 1; & \text{otherwise,} \end{cases}$$

where  $h_n^{j,l}$  is determined by (5.13). By (5.3), we know that  $h_n^{j,l} \rightarrow 0$ , therefore as  $n \rightarrow +\infty$

$$\widetilde{\psi}_n^j \rightarrow \widehat{\psi}^j, \quad a.e. \ x \in \mathbb{R}^3, \quad \text{and} \quad \widetilde{\psi}_n^j \rightarrow \widehat{\psi}^j, \quad \text{in } L_x^6,$$

which implies that

$$\left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) - F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widetilde{\psi}_n^j \right) \right| \longrightarrow 0, \\ \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widehat{\psi}^j) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widetilde{\psi}_n^j) \right| \longrightarrow 0.$$

On the other hand, by the support property of  $\widetilde{\psi}_n^j$ , we know that

$$F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widetilde{\psi}_n^j \right) = \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widetilde{\psi}_n^j).$$

Therefore, we have

$$\left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 (h_n^j T_n^j \widehat{\psi}^j) \right|$$

$$\begin{aligned}
&\leq \left| F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widehat{\psi}^j \right) - F_1 \left( \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} h_n^j T_n^j \widetilde{\psi}_n^j \right) \right| \\
&\quad + \left| \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 \left( h_n^j T_n^j \widehat{\psi}^j \right) - \sum_{j < k, \tau_n^j \rightarrow \tau_\infty^j} F_1 \left( h_n^j T_n^j \widetilde{\psi}_n^j \right) \right| \rightarrow 0.
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 5.4.** *Let  $k \in \mathbb{N}$  and radial functions  $\varphi_0, \dots, \varphi_k \in H^1(\mathbb{R}^3)$ ,  $m$  be determined by (1.5). Assume that there exist some  $\delta, \varepsilon > 0$  with  $4\varepsilon < 3\delta$  such that*

$$\sum_{j=0}^k E(\varphi_j) - \varepsilon \leq E \left( \sum_{j=0}^k \varphi_j \right) < m - \delta, \quad \text{and} \quad -\varepsilon \leq K \left( \sum_{j=0}^k \varphi_j \right) \leq \sum_{j=0}^k K(\varphi_j) + \varepsilon.$$

*Then  $\varphi_j \in \mathcal{K}^+$  for all  $j = 0, \dots, k$ .*

*Proof.* Suppose that  $K(\varphi_l) < 0$  for some  $l$ . Then by Lemma 2.9, we have

$$H(\varphi_l) \geq \inf \{ H(\varphi) \mid \varphi \in H^1(\mathbb{R}^3), \varphi \neq 0, K(\varphi) \leq 0 \} = m.$$

By the nonnegativity of  $H(\varphi_j)$  for  $j \geq 0$ , we have

$$\begin{aligned}
m \leq H(\varphi_l) &\leq \sum_{j=0}^k H(\varphi_j) = \sum_{j=0}^k \left( E(\varphi_j) - \frac{1}{6} K(\varphi_j) \right) \\
&\leq E \left( \sum_{j=0}^k \varphi_j \right) + \varepsilon - \frac{1}{6} K \left( \sum_{j=0}^k \varphi_j \right) + \frac{1}{6} \varepsilon \\
&\leq m - \delta + \varepsilon + \frac{1}{3} \varepsilon < m.
\end{aligned}$$

It is a contradiction. Hence for any  $j \in \{0, \dots, k\}$ , we have

$$K(\varphi_j) \geq 0,$$

which implies that

$$E(\varphi_j) = H(\varphi_j) + \frac{1}{6} K(\varphi_j) \geq 0,$$

and

$$E(\varphi_j) \leq \sum_{i=0}^k E(\varphi_i) < m - \delta + \varepsilon < m,$$

which means that  $\varphi_j \in \mathcal{K}^+$  for all  $j$ .  $\square$

According to the above results, we conclude as following.

**Proposition 5.5.** *Let  $\vec{v}_n(t, x)$  be a sequence of the radial solutions of the free Schrödinger equation satisfying*

$$v_n(0) \in \mathcal{K}^+ \quad \text{and} \quad E(v_n(0)) < m.$$

*Let*

$$\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x),$$

*be the linear profile decomposition given by Proposition 5.1. Then for large  $n$  and all  $j < K$ , we have*

$$v_n^j(0) \in \mathcal{K}^+, \quad w_n^K(0) \in \mathcal{K}^+,$$

*and*

$$\begin{aligned} \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| M(v_n(0)) - \sum_{j < k} M(v_n^j(0)) - M(w_n^k(0)) \right| &= 0, \\ \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| E(v_n(0)) - \sum_{j < k} E(v_n^j(0)) - E(w_n^k(0)) \right| &= 0, \\ \lim_{k \rightarrow K} \overline{\lim}_{n \rightarrow +\infty} \left| K(v_n(0)) - \sum_{j < k} K(v_n^j(0)) - K(w_n^k(0)) \right| &= 0. \end{aligned}$$

*Moreover for all  $j < K$ , we have*

$$0 \leq \underline{\lim}_{n \rightarrow +\infty} E(v_n^j(0)) \leq \overline{\lim}_{n \rightarrow +\infty} E(v_n^j(0)) \leq \overline{\lim}_{n \rightarrow +\infty} E(v_n(0)),$$

*where the last inequality becomes equality only if  $K = 1$  and  $w_n^1 \rightarrow 0$  in  $L_t^\infty \dot{H}_x^1$ .*

**5.2. Nonlinear profile decomposition.** After the linear profile decomposition of a sequence of initial data in the last subsection, we now show the nonlinear profile decomposition of a sequence of radial solutions of (1.1) with the same initial data in the energy space  $H^1(\mathbb{R}^3)$ . First we introduce some notation

$$\langle \nabla \rangle_n^j = \sqrt{(h_n^j)^2 - \Delta}, \quad \langle \nabla \rangle_\infty^j = \sqrt{(h_\infty^j)^2 - \Delta}.$$

Now let  $v_n(t, x)$  be a sequence of radial solutions for the free Schrödinger equation with initial data in  $\mathcal{K}^+$ , that is,  $v_n \in H^1(\mathbb{R}^3)$  is radial and

$$(i\partial_t + \Delta) v_n = 0, \quad v_n(0) \in \mathcal{K}^+.$$

Let

$$\vec{v}_n(t, x) = \langle \nabla \rangle v_n(t, x),$$

then by Proposition 5.1, we have a sequence of the radial, free concentrating wave  $\vec{v}_n^j(t, x)$  with  $\vec{v}_n^j(t_n^j) = T_n^j \varphi^j$ ,  $v_n^j(0) \in \mathcal{K}^+$  for  $j = 0, \dots, K$ , such that

$$\begin{aligned} \vec{v}_n(t, x) &= \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x) = \sum_{j=0}^{k-1} e^{i(t-t_n^j)\Delta} T_n^j \varphi^j + \vec{w}_n^k \\ &= \sum_{j=0}^{k-1} T_n^j e^{i\left(\frac{t-t_n^j}{(h_n^j)^2}\right)\Delta} \varphi^j + \vec{w}_n^k. \end{aligned}$$

Now for any concentrating wave  $\vec{v}_n^j$ ,  $j = 0, \dots, K$ , we undo the group action, i.e., the scaling transformation  $T_n^j$ , to look for the linear profile  $V^j$ . Let

$$\vec{v}_n^j(t, x) = T_n^j \vec{V}^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right),$$

then we have

$$(i\partial_t + \Delta) \vec{V}^j = 0, \quad \vec{V}^j(0) = \varphi^j.$$

Now let  $u_n^j(t, x)$  be the nonlinear solution of (1.1) with initial data  $v_n^j(0)$ , that is

$$\begin{aligned} (i\partial_t + \Delta) \vec{u}_n^j(t, x) &= \langle \nabla \rangle f_1(\langle \nabla \rangle^{-1} \vec{u}_n^j) + \langle \nabla \rangle f_2(\langle \nabla \rangle^{-1} \vec{u}_n^j), \\ \vec{u}_n^j(0) &= \vec{v}_n^j(0) = T_n^j \vec{V}^j(\tau_n^j), \quad u_n^j(0) \in \mathcal{K}^+, \end{aligned}$$

where  $\tau_n^j = -t_n^j/(h_n^j)^2$ . In order to look for the nonlinear profile  $\vec{U}_\infty^j$  associated to the radial, free concentrating wave  $(\vec{v}_n^j; h_n^j, t_n^j)$ , we also need undo the group action. We denote

$$\vec{u}_n^j(t, x) = T_n^j \vec{U}_n^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right),$$

then we have

$$\begin{aligned} (i\partial_t + \Delta) \vec{U}_n^j &= \left( \langle \nabla \rangle_n^j \right) f_1 \left( \left( \langle \nabla \rangle_n^j \right)^{-1} \vec{U}_n^j \right) + h_n^j \cdot \left( \langle \nabla \rangle_n^j \right) f_2 \left( \left( \langle \nabla \rangle_n^j \right)^{-1} \vec{U}_n^j \right), \\ \vec{U}_n^j(\tau_n^j) &= \vec{V}^j(\tau_n^j). \end{aligned}$$

Up to a subsequence, we may assume that there exist  $h_\infty^j \in \{0, 1\}$  and  $\tau_\infty^j \in [-\infty, \infty]$  for every  $j = \{0, \dots, K\}$ , such that

$$h_n^j \rightarrow h_\infty^j, \quad \text{and} \quad \tau_n^j \rightarrow \tau_\infty^j.$$

As  $n \rightarrow +\infty$ , the limit equation of  $\vec{U}_n^j$  is given by

$$(i\partial_t + \Delta) \vec{U}_\infty^j = \left( \langle \nabla \rangle_\infty^j \right) f_1 \left( \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{U}_\infty^j \right) + h_\infty^j \cdot \left( \langle \nabla \rangle_\infty^j \right) f_2 \left( \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{U}_\infty^j \right),$$

$$\vec{U}_\infty^j(\tau_\infty^j) = \vec{V}^j(\tau_\infty^j) \in L^2(\mathbb{R}^3).$$

Let

$$\widehat{U}_\infty^j := \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{U}_\infty^j,$$

then

$$(i\partial_t + \Delta) \widehat{U}_\infty^j = f_1 \left( \widehat{U}_\infty^j \right) + h_\infty^j \cdot f_2 \left( \widehat{U}_\infty^j \right), \quad (5.17)$$

$$\widehat{U}_\infty^j(\tau_\infty^j) = \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{V}^j(\tau_\infty^j). \quad (5.18)$$

The unique existence of a local radial solution  $\vec{U}_\infty^j$  around  $\tau_\infty^j$  is known in all cases, including  $h_\infty^j = 0$  and  $\tau_\infty^j = \pm\infty$ .  $\vec{U}_\infty^j$  on the maximal existence interval is called the nonlinear profile associated with the radial, free concentrating wave  $(\vec{v}_n^j; h_n^j, t_n^j)$ .

The nonlinear concentrating wave  $u_{(n)}^j$  associated with  $(\vec{v}_n^j; h_n^j, t_n^j)$  is defined by

$$\vec{u}_{(n)}^j(t, x) = T_n^j \vec{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right),$$

then we have

$$\begin{aligned} (i\partial_t + \Delta) \vec{u}_{(n)}^j &= \left( \sqrt{|\nabla|^2 + \left( \frac{h_\infty^j}{h_n^j} \right)^2} \right) f_1 \left( \left( \sqrt{|\nabla|^2 + \left( \frac{h_\infty^j}{h_n^j} \right)^2} \right)^{-1} \vec{u}_{(n)}^j \right) \\ &\quad + \frac{h_\infty^j}{h_n^j} \cdot \left( \sqrt{|\nabla|^2 + \left( \frac{h_\infty^j}{h_n^j} \right)^2} \right) f_2 \left( \left( \sqrt{|\nabla|^2 + \left( \frac{h_\infty^j}{h_n^j} \right)^2} \right)^{-1} \vec{u}_{(n)}^j \right) \\ &= \langle \nabla \rangle_\infty^j f_1 \left( \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{u}_{(n)}^j \right) + h_\infty^j \cdot \langle \nabla \rangle_\infty^j f_2 \left( \left( \langle \nabla \rangle_\infty^j \right)^{-1} \vec{u}_{(n)}^j \right), \\ \vec{u}_{(n)}^j(0) &= T_n^j \vec{U}_\infty^j(\tau_n^j), \end{aligned}$$

which implies that

$$\begin{aligned} \|\vec{u}_{(n)}^j(0) - \vec{u}_n^j(0)\|_{L^2} &= \|T_n^j \vec{U}_\infty^j(\tau_n^j) - T_n^j \vec{V}^j(\tau_n^j)\|_{L^2} = \|\vec{U}_\infty^j(\tau_n^j) - \vec{V}^j(\tau_n^j)\|_{L^2} \\ &\leq \|\vec{U}_\infty^j(\tau_n^j) - \vec{U}_\infty^j(\tau_\infty^j)\|_{L^2} + \|\vec{V}^j(\tau_n^j) - \vec{V}^j(\tau_\infty^j)\|_{L^2} \rightarrow 0. \end{aligned}$$



We denote

$$\vec{u}_{(n)}^j = \langle \nabla \rangle u_{(n)}^j.$$

If  $h_\infty^j = 1$ , we have  $h_n^j \equiv 1$ , then  $u_{(n)}^j \in H^1(\mathbb{R}^3)$  is radial and satisfies

$$(i\partial_t + \Delta) u_{(n)}^j = f_1(u_{(n)}^j) + f_2(u_{(n)}^j).$$

If  $h_\infty^j = 0$ , then  $u_{(n)}^j \in H^1(\mathbb{R}^3)$  is radial and satisfies

$$(i\partial_t + \Delta) u_{(n)}^j = \frac{|\nabla|}{\langle \nabla \rangle} f_1 \left( \frac{\langle \nabla \rangle}{|\nabla|} u_{(n)}^j \right).$$

Let  $u_n$  be a sequence of (local) radial solutions of (1.1) with initial data in  $\mathcal{K}^+$  at  $t = 0$ , and let  $v_n$  be the sequence of the radial, free solutions with the same initial data. We consider the linear profile decomposition given by Proposition 5.1

$$\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x), \quad \vec{v}_n^j(t_n^j) = T_n^j \varphi^j, \quad v_n^j(0) \in \mathcal{K}^+.$$

With each free concentrating wave  $\{\vec{v}_n^j\}_{n \in \mathbb{N}}$ , we associate the nonlinear concentrating wave  $\{\vec{u}_{(n)}^j\}_{n \in \mathbb{N}}$ . A nonlinear profile decomposition of  $u_n$  is given by

$$\vec{u}_{(n)}^{<k}(t, x) := \sum_{j=0}^{k-1} \vec{u}_{(n)}^j(t, x) = \sum_{j=0}^{k-1} T_n^j \vec{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right). \quad (5.19)$$

Since the smallness condition (5.2) and the orthogonality condition (5.3) ensure that every nonlinear concentrating wave and the remainder interacts weakly with the others, we will show that  $\vec{u}_{(n)}^{<k} + \vec{w}_n^k$  is a good approximation for  $\vec{u}_n$  provided that each nonlinear profile has the finite global Strichartz norm.

Now we define the Strichartz norms for the nonlinear profile decomposition. Let  $ST(I)$  and  $ST^*(I)$  be the function spaces on  $I \times \mathbb{R}^3$  defined as Section 4

$$\begin{aligned} ST(I) &:= \left( L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9 \cap L_t^6 \dot{B}_{18/7,2}^{1/2} \cap L_{t,x}^5 \right) (I \times \mathbb{R}^3), \\ ST^*(I) &:= \left( L_t^2 \dot{B}_{18/11,2}^{1/3} \cap L_t^2 \dot{B}_{6/5,2}^{1/2} \right) (I \times \mathbb{R}^3). \end{aligned}$$

The Strichartz norm for the nonlinear profile  $\widehat{U}_\infty^j$  depends on the scaling  $h_\infty^j$ .

$$ST_\infty^j(I) := \begin{cases} ST(I), & \text{for } h_\infty^j = 1, \\ \left( L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9 \right) (I \times \mathbb{R}^3), & \text{for } h_\infty^j = 0. \end{cases}$$

**Lemma 5.6.** *In the nonlinear profile decomposition (5.19). Suppose that for each  $j < K$ , we have*

$$\|\widehat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})} + \|\vec{U}_\infty^j\|_{L_t^\infty L_x^2(\mathbb{R}^3)} < \infty.$$

*Then for any finite interval  $I$ , any  $j < K$  and any  $k \leq K$ , we have*

$$\overline{\lim}_{n \rightarrow +\infty} \|u_{(n)}^j\|_{ST(I)} \lesssim \|\widehat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})}, \quad (5.20)$$

$$\overline{\lim}_{n \rightarrow +\infty} \|u_{(n)}^{<k}\|_{ST(I)}^2 \lesssim \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k} \|u_{(n)}^j\|_{ST(I)}^2, \quad (5.21)$$

*where the implicit constants do not depend on  $I, j$  or  $k$ . We also have*

$$\lim_{n \rightarrow +\infty} \left\| f_1 \left( u_{(n)}^{<k} \right) - \sum_{j < k} \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_1 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) \right\|_{ST^*(I)} = 0, \quad (5.22)$$

$$\lim_{n \rightarrow +\infty} \left\| f_2 \left( u_{(n)}^{<k} \right) - \sum_{j < k} h_\infty^j \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_2 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) \right\|_{ST^*(I)} = 0. \quad (5.23)$$

*Proof. Proof of (5.20).* By the definitions of  $u_{(n)}^j$  and  $\widehat{U}_\infty^j$ , we know that

$$\begin{aligned} u_{(n)}^j(t, x) &= \langle \nabla \rangle^{-1} \vec{u}_{(n)}^j(t, x) = \langle \nabla \rangle^{-1} T_n^j \vec{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right) \\ &= \langle \nabla \rangle^{-1} T_n^j \langle \nabla \rangle_\infty^j \widehat{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right) = h_n^j T_n^j \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle_n^j} \widehat{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right). \end{aligned}$$

For the case  $h_\infty^j = 1$ , we have  $u_{(n)}^j(t, x) = \widehat{U}_\infty^j(t - t_n^j, x)$ , hence (5.20) is trivial. For the case  $h_\infty^j = 0$ , by the above relation between  $u_{(n)}^j$  and  $\widehat{U}_\infty^j$ , we have

$$\begin{aligned} \|u_{(n)}^j\|_{(L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9)(I \times \mathbb{R}^3)} &\lesssim \left\| \frac{|\nabla|}{\langle \nabla \rangle_n^j} \widehat{U}_\infty^j \right\|_{(L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9)(\mathbb{R} \times \mathbb{R}^3)} \\ &\lesssim \|\widehat{U}_\infty^j\|_{(L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9)(\mathbb{R} \times \mathbb{R}^3)}, \end{aligned}$$

and

$$\begin{aligned} \|u_{(n)}^j\|_{L_t^6 \dot{B}_{18/7,2}^{1/2}(I \times \mathbb{R}^3)} &\lesssim |I|^{\frac{1}{12}} \|u_{(n)}^j\|_{L_t^{12} \dot{B}_{18/7,2}^{1/2}} \lesssim |I|^{\frac{1}{12}} (h_n^j)^{\frac{1}{3}} \|\widehat{U}_\infty^j\|_{L_t^{12} \dot{B}_{18/7,2}^{\frac{5}{6}}} \rightarrow 0, \\ \|u_{(n)}^j\|_{L_{t,x}^5(I \times \mathbb{R}^3)} &\lesssim |I|^{\frac{7}{60}} \|u_{(n)}^j\|_{L_t^{12} L_x^5} \lesssim |I|^{\frac{7}{60}} (h_n^j)^{\frac{4}{15}} \|\widehat{U}_\infty^j\|_{L_t^{12} \dot{B}_{5,2}^{\frac{4}{15}}} \rightarrow 0. \end{aligned}$$

where we use the fact that the boundedness of  $\widehat{U}_\infty^j$  in  $L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9 \cap L^\infty \dot{H}^1$  implies its boundedness in  $L_t^{12} \dot{B}_{18/7,2}^{\frac{5}{6}} \cap L_t^{12} \dot{B}_{5,2}^{\frac{4}{15}}$  by (5.17).

**Proof of (5.21).** We estimate the left hand side of (5.21) by

$$\begin{aligned} \|u_{(n)}^{\leq k}\|_{ST(I)}^2 &= \left\| \sum_{j < k, h_\infty^j = 1} u_{(n)}^j + \sum_{j < k, h_\infty^j = 0} u_{(n)}^j \right\|_{ST(I)}^2 \\ &\lesssim \left\| \sum_{j < k, h_\infty^j = 1} u_{(n)}^j \right\|_{ST(I)}^2 + \left\| \sum_{j < k, h_\infty^j = 0} u_{(n)}^j \right\|_{ST(I)}^2. \end{aligned}$$

For the case  $h_\infty^j = 1$ . Define  $\widehat{U}_{\infty,R}^j$  and  $u_{(n),R}^j$  by

$$\widehat{U}_{\infty,R}^j(t, x) = \chi_R(t, x) \widehat{U}_\infty^j(t, x), \quad u_{(n),R}^j(t, x) = T_n^j \widehat{U}_{\infty,R}^j(t - t_n^j),$$

where  $\chi_R$  is the cut-off function as in Remark 1.6. Then we have

$$\left\| \sum_{j < k, h_\infty^j = 1} u_{(n)}^j \right\|_{ST(I)}^2 \lesssim \left\| \sum_{j < k, h_\infty^j = 1} u_{(n),R}^j \right\|_{ST(I)}^2 + \left\| \sum_{j < k, h_\infty^j = 1} u_{(n)}^j - \sum_{j < k, h_\infty^j = 1} u_{(n),R}^j \right\|_{ST(I)}^2.$$

On one hand, we know that

$$\left\| \sum_{j < k, h_\infty^j = 1} u_{(n)}^j - \sum_{j < k, h_\infty^j = 1} u_{(n),R}^j \right\|_{ST(I)} \leq \sum_{j < k, h_\infty^j = 1} \|(1 - \chi_R) u_{(n)}^j\|_{ST(I)} \rightarrow 0,$$

as  $R \rightarrow +\infty$ . On the other hand, by (5.3) and the similar orthogonality analysis as in [17], we know that

$$\overline{\lim}_{n \rightarrow +\infty} \left\| \sum_{j < k, h_\infty^j = 1} u_{(n),R}^j \right\|_{ST(I)}^2 \lesssim \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k, h_\infty^j = 1} \|u_{(n),R}^j\|_{ST(I)}^2 \lesssim \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k, h_\infty^j = 1} \|u_{(n)}^j\|_{ST(I)}^2.$$

For the case  $h_\infty^j = 0$ , On one hand, by  $h_n^j \rightarrow 0$ , we have

$$\overline{\lim}_{n \rightarrow +\infty} \left\| \sum_{j < k, h_\infty^j = 0} u_{(n)}^j \right\|_{\left(L_t^6 \dot{B}_{18/7,2}^{1/2} \cap L_{t,x}^5\right)(I \times \mathbb{R}^3)} = 0.$$

On the other hand, by (5.3) and the analogue approximation analysis as in [17], we have

$$\overline{\lim}_{n \rightarrow +\infty} \left\| \sum_{j < k, h_\infty^j = 0} u_{(n)}^j \right\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}(I \times \mathbb{R}^3)}^2 \lesssim \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k, h_\infty^j = 0} \|u_{(n)}^j\|_{L_t^{10} \dot{B}_{90/19,2}^{1/3}(I \times \mathbb{R}^3)}^2,$$

$$\overline{\lim}_{n \rightarrow +\infty} \left\| \sum_{j < k, h_\infty^j = 0} u_{(n)}^j \right\|_{L_t^{12} L_x^9(I \times \mathbb{R}^3)}^2 \lesssim \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k, h_\infty^j = 0} \|u_{(n)}^j\|_{L_t^{12} L_x^9(I \times \mathbb{R}^3)}^2.$$

**Proof of (5.22).** Let  $u_{<n>}^k(t, x) := \sum_{j < k} u_{<n>}^j(t, x)$  where

$$u_{<n>}^j(t, x) := \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j = \frac{1}{\langle \nabla \rangle_\infty^j} \vec{u}_{(n)}^j = \frac{1}{\langle \nabla \rangle_\infty^j} T_n^j \vec{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right) = h_n^j T_n^j \widehat{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right),$$

and

$$u_{(n)}^j(t, x) = h_n^j T_n^j \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle_n^j} \widehat{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right).$$

Then we have

$$\begin{aligned} & \left\| f_1(u_{(n)}^{<k>}) - \sum_{j < k} \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle_\infty^j} f_1 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) \right\|_{ST^*} \\ & \leq \left\| f_1(u_{(n)}^{<k>}) - f_1(u_{<n>}^{<k>}) \right\|_{ST^*} + \left\| f_1(u_{<n>}^{<k>}) - \sum_{j < k} f_1(u_{<n>}^j) \right\|_{ST^*} \\ & \quad + \left\| \sum_{j < k} f_1(u_{<n>}^j) - \sum_{j < k} \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle_\infty^j} f_1(u_{<n>}^j) \right\|_{ST^*} \\ & \leq \left\| f_1(u_{(n)}^{<k>}) - f_1(u_{<n>}^{<k>}) \right\|_{ST^*} + \left\| f_1(u_{<n>}^{<k>}) - \sum_{j < k} f_1(u_{<n>}^j) \right\|_{ST^*} \\ & \quad + \left\| \sum_{j < k, h_\infty^j = 0} f_1(u_{<n>}^j) - \sum_{j < k, h_\infty^j = 0} \frac{|\nabla|}{\langle \nabla \rangle} f_1(u_{<n>}^j) \right\|_{ST^*}. \end{aligned}$$

By (5.3) and the approximation argument in [17], we have

$$\left\| f_1(u_{(n)}^{<k>}) - f_1(u_{<n>}^{<k>}) \right\|_{ST^*} + \left\| f_1(u_{<n>}^{<k>}) - \sum_{j < k} f_1(u_{<n>}^j) \right\|_{ST^*} \longrightarrow 0$$

as  $n \rightarrow +\infty$ . In addition, by  $h_n^j \rightarrow 0$  as  $n \rightarrow +\infty$ , we have

$$\begin{aligned} & \left\| \sum_{j < k, h_\infty^j = 0} \left( 1 - \frac{|\nabla|}{\langle \nabla \rangle} \right) f_1(u_{<n>}^j) \right\|_{L_t^2 \dot{B}_{18/11,2}^{1/3}} = \left\| \left( 1 - \frac{|\nabla|}{\langle \nabla \rangle_n^j} \right) \sum_{j < k, h_\infty^j = 0} f_1(\widehat{U}_\infty^j) \right\|_{L_t^2 \dot{B}_{18/11,2}^{1/3}} \longrightarrow 0, \\ & \left\| \sum_{j < k, h_\infty^j = 0} \left( 1 - \frac{|\nabla|}{\langle \nabla \rangle} \right) f_1(u_{<n>}^j) \right\|_{L_t^2 \dot{B}_{6/5,2}^{1/2}} = (h_n^j)^{1/2} \left\| \left( 1 - \frac{|\nabla|}{\langle \nabla \rangle_n^j} \right) \sum_{j < k, h_\infty^j = 0} f_1(\widehat{U}_\infty^j) \right\|_{L_t^2 \dot{B}_{6/5,2}^{1/2}} \longrightarrow 0, \end{aligned}$$

as  $n \rightarrow +\infty$ . Therefore, we have

$$\lim_{n \rightarrow +\infty} \left\| f_1 \left( u_{(n)}^{<k} \right) - \sum_{j < k} \frac{\langle \nabla \rangle^j}{\langle \nabla \rangle} f_1 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle^j} u_{(n)}^j \right) \right\|_{ST^*} = 0.$$

**Proof of (5.23).** Note that

$$\begin{aligned} & \left\| f_2 \left( u_{(n)}^{<k} \right) - \sum_{j < k} h_\infty^j \frac{\langle \nabla \rangle^j}{\langle \nabla \rangle} f_2 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle^j} u_{(n)}^j \right) \right\|_{ST^*} \\ & \leq \left\| f_2 \left( u_{(n)}^{<k} \right) - f_2 \left( u_{\langle n \rangle}^{<k} \right) \right\|_{ST^*} + \left\| f_2 \left( u_{\langle n \rangle}^{<k} \right) - \sum_{j < k} f_2 \left( u_{\langle n \rangle}^j \right) \right\|_{ST^*} + \left\| \sum_{j < k, h_\infty^j = 0} f_2 \left( u_{\langle n \rangle}^j \right) \right\|_{ST^*}. \end{aligned}$$

By the analogue analysis, we have

$$\left\| f_2 \left( u_{(n)}^{<k} \right) - f_2 \left( u_{\langle n \rangle}^{<k} \right) \right\|_{ST^*} + \left\| f_2 \left( u_{\langle n \rangle}^{<k} \right) - \sum_{j < k} f_2 \left( u_{\langle n \rangle}^j \right) \right\|_{ST^*} \longrightarrow 0,$$

and

$$\left\| \sum_{j < k, h_\infty^j = 0} f_2 \left( u_{\langle n \rangle}^j \right) \right\|_{ST^*} \longrightarrow 0$$

as  $n \rightarrow +\infty$ . Hence, we obtain

$$\lim_{n \rightarrow +\infty} \left\| f_2 \left( u_{(n)}^{<k} \right) - \sum_{j < k} h_\infty^j \frac{\langle \nabla \rangle^j}{\langle \nabla \rangle} f_2 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle^j} u_{(n)}^j \right) \right\|_{ST^*} = 0.$$

These complete the proof.  $\square$

After this preliminaries, we now show that  $\vec{u}_{(n)}^{<k} + \vec{w}_n^k$  is a good approximation for  $\vec{u}_n$  provided that each nonlinear profile has finite global Strichartz norm.

**Proposition 5.7.** *Let  $u_n$  be a sequence of local, radial solutions of (1.1) around  $t = 0$  in  $\mathcal{K}^+$  satisfying*

$$M(u_n) < \infty, \quad \overline{\lim}_{n \rightarrow \infty} E(u_n) < m.$$

*Suppose that in the nonlinear profile decomposition (5.19), every nonlinear profile  $\widehat{U}_\infty^j$  has finite global Strichartz and energy norms we have*

$$\|\widehat{U}_\infty^j\|_{ST_\infty^j(\mathbb{R})} + \|\vec{U}_\infty^j\|_{L_t^\infty L_x^2(\mathbb{R}^3)} < \infty.$$

Then  $u_n$  is bounded for large  $n$  in the Strichartz and the energy norms

$$\lim_{n \rightarrow \infty} \|u_n\|_{ST(\mathbb{R})} + \|\vec{u}_n\|_{L_t^\infty L_x^2(\mathbb{R})} < \infty.$$

*Proof.* We only need to verify the condition of Proposition 4.1. Note that  $u_{(n)}^{<k} + w_n^k$  satisfies that

$$\begin{aligned} (i\partial_t + \Delta) \left( u_{(n)}^{<k} + w_n^k \right) &= f_1 \left( u_{(n)}^{<k} + w_n^k \right) + f_2 \left( u_{(n)}^{<k} + w_n^k \right) \\ &\quad + f_1 \left( u_{(n)}^{<k} \right) - f_1 \left( u_{(n)}^{<k} + w_n^k \right) + f_2 \left( u_{(n)}^{<k} \right) - f_2 \left( u_{(n)}^{<k} + w_n^k \right) \\ &\quad + \sum_{j < k} \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_1 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) - f_1 \left( u_{(n)}^{<k} \right) + \sum_{j < k} h_\infty^j \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_2 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) - f_2 \left( u_{(n)}^{<k} \right). \end{aligned}$$

First, by the construction of  $\vec{u}_{(n)}^{<k}$ , we know that

$$\left\| \left( \vec{u}_{(n)}^{<k}(0) + \vec{w}_n^k(0) \right) - \vec{u}_n(0) \right\|_{L_x^2} \leq \sum_{j < k} \left\| \vec{u}_{(n)}^j(0) - \vec{u}_n^j(0) \right\|_{L_x^2} \rightarrow 0,$$

as  $n \rightarrow +\infty$ , which also implies that for large  $n$ , we have

$$\left\| \vec{u}_{(n)}^{<k}(0) + \vec{w}_n^k(0) \right\|_{L_x^2} \leq E_0.$$

Next, by the linear profile decomposition in Proposition 5.1, we know that

$$\begin{aligned} \|u_n(0)\|_{L^2}^2 &= \|v_n(0)\|_{L_x^2}^2 = \sum_{j < k} \|v_n^j(0)\|_{L_x^2}^2 + \|w_n^k(0)\|_{L_x^2}^2 + o_n(1) \\ &\geq \sum_{j < k} \|v_n^j(0)\|_{L_x^2}^2 + o_n(1) = \sum_{j < k} \|u_{(n)}^j(0)\|_{L_x^2}^2 + o_n(1), \\ \|u_n(0)\|_{\dot{H}_x^1}^2 &= \|v_n(0)\|_{\dot{H}_x^1}^2 = \sum_{j < k} \|v_n^j(0)\|_{\dot{H}_x^1}^2 + \|w_n^k(0)\|_{\dot{H}_x^1}^2 + o_n(1) \\ &\geq \sum_{j < k} \|v_n^j(0)\|_{\dot{H}_x^1}^2 + o_n(1) = \sum_{j < k} \|u_{(n)}^j(0)\|_{\dot{H}_x^1}^2 + o_n(1), \end{aligned}$$

which means except for a finite set  $J \subset \mathbb{N}$ , the energy of  $u_{(n)}^j$  with  $j \notin J$  is smaller than the iteration threshold, hence we have

$$\|u_{(n)}^j\|_{ST(\mathbb{R})} \lesssim \|\vec{u}_{(n)}^j(0)\|_{L_x^2},$$

thus, for any finite interval  $I$ , by Lemma 5.6, we have

$$\sup_k \lim_{n \rightarrow +\infty} \|u_{(n)}^{<k}\|_{ST(I)}^2 \lesssim \sup_k \lim_{n \rightarrow +\infty} \sum_{j < k} \|u_{(n)}^j\|_{ST(I)}^2$$

$$\begin{aligned}
&= \sup_k \overline{\lim}_{n \rightarrow +\infty} \left[ \sum_{j < k, j \in J} \|u_{(n)}^j\|_{ST(I)}^2 + \sum_{j < k, j \notin J} \|u_{(n)}^j\|_{ST(I)}^2 \right] \\
&\lesssim \sum_{j < k, j \in J} \|\widehat{U}_\infty^j\|_{ST_\infty^j(I)}^2 + \sup_k \overline{\lim}_{n \rightarrow +\infty} \sum_{j < k, j \notin J} \|\vec{u}_{(n)}^j(0)\|_{L_x^2}^2 \\
&< \infty.
\end{aligned}$$

This together with the Strichartz estimate for  $w_n^k$  implies that

$$\sup_k \overline{\lim}_{n \rightarrow +\infty} \|u_{(n)}^{<k} + w_n^k\|_{ST(I)}^2 < \infty.$$

Last we need show the nonlinear perturbation is small in some sense. By Proposition 5.1 and Lemma 5.6, we have

$$\begin{aligned}
&\|f_1(u_{(n)}^{<k}) - f_1(u_{(n)}^{<k} + w_n^k)\|_{ST^*(I)} \rightarrow 0, \\
&\|f_2(u_{(n)}^{<k}) - f_2(u_{(n)}^{<k} + w_n^k)\|_{ST^*(I)} \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \sum_{j < k} \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_1 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) - f_1(u_{(n)}^{<k}) \right\|_{ST^*(I)} \rightarrow 0, \\
&\left\| \sum_{j < k} h_\infty^j \frac{\langle \nabla \rangle_\infty^j}{\langle \nabla \rangle} f_2 \left( \frac{\langle \nabla \rangle}{\langle \nabla \rangle_\infty^j} u_{(n)}^j \right) - f_2(u_{(n)}^{<k}) \right\|_{ST^*(I)} \rightarrow 0,
\end{aligned}$$

as  $n \rightarrow +\infty$ . Therefore, by Proposition 4.1, we can obtain the desired result, which concludes the proof.  $\square$

## 6. PART II: GWP AND SCATTERING FOR $\mathcal{K}^+$

After the stability analysis of the scattering solution of (1.1) and the compactness analysis (linear and nonlinear profile decompositions) of a sequence of the radial solutions of (1.1) in the energy space. We now use them to show the scattering result of Theorem 1.3 by contradiction.

Let  $E^*$  be the threshold for the uniform Strichartz norm bound, i.e.,

$$E^* := \sup\{A > 0, ST(A) < \infty\}$$

where  $ST(A)$  denotes the supremum of  $\|u\|_{ST(I)}$  for any strong radial solution  $u$  of (1.1) in  $\mathcal{K}^+$  on any interval  $I$  satisfying  $E(u) \leq A$ ,  $M(u) < \infty$ .

The small solution scattering theory gives us  $E^* > 0$ .

Now we are going to show that  $E^* \geq m$  by contradiction. From now on, suppose that  $E^* \geq m$  fails, that is, we assume that

$$E^* < m. \quad (6.1)$$

**6.1. Existence of a critical element.** In this subsection, by the profile decomposition and the stability theory of the scattering solution of (1.1), we show the existence of the critical element, which is the radial, energy solution of (1.1) with the smallness energy  $E^*$  and infinite Strichartz norm.

By the definition of  $E^*$  and the fact that  $E^* < m$ , there exist a sequence of radial solutions  $\{u_n\}_{n \in \mathbb{N}}$  of (1.1) in  $\mathcal{K}^+$ , which have the maximal existence interval  $I_n$  and satisfy that

$$M(u_n) < \infty, \quad E(u_n) \rightarrow E^* < m, \quad \|u_n\|_{ST(I_n)} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty,$$

then we have  $\|u_n\|_{H^1} < \infty$  by Lemma 2.12. By the compact argument (profile decomposition) and the stability theory, we can show that

**Theorem 6.1.** *Let  $u_n$  be a sequence of radial solutions of (1.1) in  $\mathcal{K}^+$  on  $I_n \subset \mathbb{R}$  satisfying*

$$M(u_n) < \infty, \quad E(u_n) \rightarrow E^* < m, \quad \|u_n\|_{ST(I_n)} \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

*Then there exists a global, radial solution  $u_c$  of (1.1) in  $\mathcal{K}^+$  satisfying*

$$E(u_c) = E^* < m, \quad K(u_c) > 0, \quad \|u_c\|_{ST(\mathbb{R})} = \infty.$$

*In addition, there are a sequence  $t_n \in \mathbb{R}$  and radial function  $\varphi \in L^2(\mathbb{R}^3)$  such that, up to a subsequence, we have as  $n \rightarrow +\infty$ ,*

$$\left\| \frac{|\nabla|}{\langle \nabla \rangle} \left( \vec{u}_n(0, x) - e^{-it_n \Delta} \varphi(x) \right) \right\|_{L^2} \rightarrow 0. \quad (6.2)$$

*Proof.* By the time translation symmetry of (1.1), we can translate  $u_n$  in  $t$  such that  $0 \in I_n$  for all  $n$ . Then by the linear and nonlinear profile decomposition of  $u_n$ , we have

$$\begin{aligned} e^{it\Delta} \vec{u}_n(0, x) &= \sum_{j < k} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x), \quad \vec{v}_n^j(t, x) = e^{i(t-t_n^j)\Delta} T_n^j \varphi^j, \\ \vec{u}_{(n)}^{<k}(t, x) &= \sum_{j < k} \vec{u}_{(n)}^j(t, x), \quad \vec{u}_{(n)}^j(t, x) = T_n^j \vec{U}_\infty^j \left( \frac{t - t_n^j}{(h_n^j)^2} \right), \\ \|\vec{u}_{(n)}^j(0) - \vec{v}_n^j(0)\|_{L^2} &\rightarrow 0. \end{aligned}$$

By Proposition 5.5 and the following observations that



- (1) Every radial solution of (1.1) in  $\mathcal{K}^+$  with the energy less than  $E^*$  has global finite Strichartz norm by the definition of  $E^*$ .
- (2) Lemma 5.7 precludes that all the nonlinear profiles  $\vec{U}_\infty^j$  have finite global Strichartz norm.

we deduce that there is only one radial profile and

$$E(u_{(n)}^0(0)) \rightarrow E^*, \quad u_{(n)}^0(0) \in \mathcal{K}^+, \quad \|\widehat{U}_\infty^0\|_{ST_\infty^0(I)} = \infty, \quad \|w_n^1\|_{L_t^\infty \dot{H}_x^1} \rightarrow 0.$$

If  $h_n^0 \rightarrow 0$ , then  $\widehat{U}_\infty^0 = |\nabla|^{-1} \vec{U}_\infty^0$  solves the  $\dot{H}^1$ -critical NLS

$$(i\partial_t + \Delta) \widehat{U}_\infty^0 = f_1(\widehat{U}_\infty^0)$$

and satisfies

$$E^c(\widehat{U}_\infty^0(\tau_\infty^0)) = E^* < m, \quad K^c(\widehat{U}_\infty^0(\tau_\infty^0)) \geq 0, \quad \|\widehat{U}_\infty^0\|_{(L_t^{10} \dot{B}_{90/19,2}^{1/3} \cap L_t^{12} L_x^9)(I \times \mathbb{R}^3)} = \infty.$$

However, it is in contradiction with Kenig-Merle's result<sup>4</sup> in [19]. Hence  $h_n^0 \equiv 1$ , which implies (6.2).

Now we show that  $\widehat{U}_\infty^0 = \langle \nabla \rangle^{-1} \vec{U}_\infty^j$  is a global solution, which is the consequence of the compactness of (6.2). Suppose not, then we can choose a sequence  $t_n \in \mathbb{R}$  which approaches the maximal existence time. Since  $\widehat{U}_\infty^0(t + t_n)$  satisfies the assumption of this theorem, then applying the above argument to it, we obtain that for some  $\psi \in L^2$  and another sequence  $t'_n \in \mathbb{R}$ , as  $n \rightarrow +\infty$

$$\left\| \frac{|\nabla|}{\langle \nabla \rangle} \left( \vec{U}_\infty^0(t_n) - e^{-it'_n \Delta} \psi(x) \right) \right\|_{L^2} \rightarrow 0. \quad (6.3)$$

Let  $\vec{v}(t) := e^{it\Delta} \psi$ . For any  $\varepsilon > 0$ , there exist  $\delta > 0$  with  $I = [-\delta, \delta]$  such that

$$\|\langle \nabla \rangle^{-1} \vec{v}(t - t'_n)\|_{ST(I)} \leq \varepsilon,$$

which together with (6.3) implies that for sufficiently large  $n$

$$\|\langle \nabla \rangle^{-1} e^{it\Delta} \vec{U}_\infty^0(t_n)\|_{ST(I)} \leq \varepsilon.$$

If  $\varepsilon$  is small enough, this implies that the solution  $\widehat{U}_\infty^0$  exists on  $[t_n - \delta, t_n + \delta]$  for large  $n$  by the small data theory. This contradicts the choice of  $t_n$ . Hence  $\widehat{U}_\infty^0$  is a global solution and it is just the desired critical element  $u_c$ . By Proposition 1.1, we know that  $K(u_c) > 0$ .  $\square$

---

<sup>4</sup>By the global  $L_{t,x}^{10}$  estimate of solution  $u$  of (1.2), we can obtain the global  $L_t^q \dot{W}_x^{1,r}$  estimate of  $u$  for any Schrödinger  $L^2$ -admissible pair  $(q, r)$ .

**6.2. Compactness of the critical element.** In order to preclude the critical element, we need obtain some useful properties about the critical element. In the following subsections, we establish some properties about the critical element by its minimal energy with infinite Strichartz norm, especially its compactness and its consequence. Since (1.1) is symmetric in  $t$ , we may assume that

$$\|u_c\|_{ST(0,+\infty)} = \infty, \quad (6.4)$$

we call it a forward critical element.

**Proposition 6.2.** *Let  $u_c$  be a forward critical element. Then the set*

$$\{u_c(t, x); 0 < t < \infty\}$$

*is precompact in  $\dot{H}^s$  for any  $s \in (0, 1]$ .*

*Proof.* By the conservation of the mass, it suffices to prove the precompactness of  $u_c(t_n)$  in  $\dot{H}^1$  for any positive time  $t_1, t_2, \dots$ . If  $t_n$  converges, then it is trivial from the continuity in  $t$ .

If  $t_n \rightarrow +\infty$ . Applying Theorem 6.1 to the sequence of solutions  $\vec{u}_c(t + t_n)$ , we get another sequence  $t'_n \in \mathbb{R}$  and radial function  $\varphi \in L^2$  such that

$$\frac{|\nabla|}{\langle \nabla \rangle} \left( \vec{u}_c(t_n, x) - e^{-it'_n \Delta} \varphi(x) \right) \rightarrow 0 \quad \text{in } L^2.$$

(1) If  $t'_n \rightarrow -\infty$ , then we have

$$\| \langle \nabla \rangle^{-1} e^{it\Delta} \vec{u}_c(t_n) \|_{ST(0,+\infty)} = \| \langle \nabla \rangle^{-1} e^{it\Delta} \varphi \|_{ST(-t'_n,+\infty)} + o_n(1) \rightarrow 0.$$

Hence  $u_c$  can solve (1.1) for  $t > t_n$  with large  $n$  globally by iteration with small Strichartz norms, which contradicts (6.4).

(2) If  $t'_n \rightarrow +\infty$ , then we have

$$\| \langle \nabla \rangle^{-1} e^{it\Delta} \vec{u}_c(t_n) \|_{ST(-\infty,0)} = \| \langle \nabla \rangle^{-1} e^{it\Delta} \varphi \|_{ST(-\infty,-t'_n)} + o_n(1) \rightarrow 0$$

Hence  $u_c$  can solve (1.1) for  $t < t_n$  with large  $n$  with vanishing Strichartz norms, which implies  $u_c = 0$  by taking the limit, which is a contradiction.

Thus  $t'_n$  is bounded, which implies that  $t'_n$  is precompact, so is  $u_c(t_n, x)$  in  $\dot{H}^1$ .  $\square$

As a consequence, the energy of  $u_c$  stays within a fixed radius for all positive time, modulo arbitrarily small rest. More precisely, we define the exterior energy by

$$E_R(u; t) = \int_{|x| \geq R} \left( |\nabla u(t, x)|^2 + |u(t, x)|^4 + |u(t, x)|^6 \right) dx$$

for any  $R > 0$ . Then we have

**Corollary 6.3.** *Let  $u_c$  be a forward critical element. then for any  $\varepsilon$ , there exist  $R_0(\varepsilon) > 0$  such that*

$$E_{R_0}(u_c; t) \leq \varepsilon E(u_c), \text{ for any } t > 0.$$

**6.3. Death of the critical element.** We are in a position to preclude the soliton-like solution by a truncated Virial identity.

**Theorem 6.4.** *The critical element  $u_c$  of (1.1) cannot be a soliton in the sense of Theorem 6.1.*

*Proof.* We still drop the subscript  $c$ . Now let  $\phi$  be a smooth, radial function satisfying  $0 \leq \phi \leq 1$ ,  $\phi(x) = 1$  for  $|x| \leq 1$ , and  $\phi(x) = 0$  for  $|x| \geq 2$ . For some  $R$ , we define

$$V_R(t) := \int_{\mathbb{R}^3} \phi_R(x) |u(t, x)|^2 dx, \quad \phi_R(x) = R^2 \phi\left(\frac{|x|^2}{R^2}\right).$$

On one hand, we have

$$\partial_t V_R(t) = 4\Im \int_{\mathbb{R}^3} \phi' \left( \frac{|x|^2}{R^2} \right) x \cdot \nabla u(t, x) \overline{u(t, x)} dx.$$

Therefore, we have

$$|\partial_t V_R(t)| \lesssim R \tag{6.5}$$

for all  $t \geq 0$  and  $R > 0$ .

On the other hand, by Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned} \partial_t^2 V_R(t) &= 4 \int_{\mathbb{R}^3} \phi_R''(r) |\nabla u(t, x)|^2 dx - \int_{\mathbb{R}^3} (\Delta^2 \phi_R)(x) |u(t, x)|^2 dx \\ &\quad - \frac{4}{3} \int_{\mathbb{R}^3} (\Delta \phi_R)(x) |u(t, x)|^6 dx + \int_{\mathbb{R}^3} (\Delta \phi_R)(x) |u(t, x)|^4 dx \\ &= 4 \int_{\mathbb{R}^3} \left( 2|\nabla u(t, x)|^2 - 2|u(t, x)|^6 + \frac{3}{2}|u(t, x)|^4 \right) dx \\ &\quad + O \left( \int_{|x| \geq R} (|\nabla u(t, x)|^2 + |u(t, x)|^6 + |u(t, x)|^4) dx + \left( \int_{R \leq |x| \leq 2R} |u(t, x)|^6 dx \right)^{1/3} \right) \\ &= 4K(u(t)) + O \left( \int_{|x| \geq R} (|\nabla u(t, x)|^2 + |u(t, x)|^4) dx + \left( \int_{R \leq |x| \leq 2R} |u(t, x)|^6 dx \right)^{1/3} \right). \end{aligned}$$

By Lemma 2.13, we have

$$4K(u(t)) = 4 \int_{\mathbb{R}^3} \left( 2|\nabla u(t, x)|^2 - 2|u(t, x)|^6 + \frac{3}{2}|u(t, x)|^4 \right) dx$$

$$\begin{aligned}
&\gtrsim \min \left( 6(m - E(u(t))), \frac{2}{3} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|u(t)\|_{L^4}^4 \right) \\
&\gtrsim \|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^4}^4 \\
&\gtrsim E(u(t)),
\end{aligned}$$

Thus, choosing  $\eta > 0$  sufficiently small and  $R := C(\eta)$  and by Corollary 6.3, we obtain

$$\partial_t^2 V_R(t) \gtrsim E(u(t)) = E(u_0),$$

which implies that for all  $T_1 > T_0$

$$(T_1 - T_0)E(u_0) \lesssim R = C(\eta).$$

Taking  $T_1$  sufficiently large, we obtain a contradiction unless  $u \equiv 0$ . But  $u \equiv 0$  is not consistent with the fact that  $\|u\|_{ST(\mathbb{R})} = \infty$ .  $\square$

## REFERENCES

- [1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Diff. Geom., **11** (1976), 573–598.
- [2] H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math., **121** (1999), 131–175.
- [3] J. Bourgain, *Global well-posedness of defocusing 3D critical NLS in the radial case*, J. Amer. Math. Soc., **12**(1999), 145–171.
- [4] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, Amer. Math. Soc. Colloq. Publ. **46**, Amer. Math. Soc., Providence, 1999.
- [5] T. Cazenave, *Semilinear Schrödinger equations*. Courant Lecture Notes in Mathematics, Vol. **10**. New York: New York University Courant Institute of Mathematical Sciences, 2003.
- [6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for rough solution of a nonlinear Schrödinger equation on  $\mathbb{R}^3$* , Comm. Pure Appl. Math., **57**(2004), 987–1014.
- [7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in  $\mathbb{R}^3$* , Ann. of Math., **166**(2007), 1–100.
- [8] B. Dodson, *Global well-posedness and scattering for the defocusing,  $L^2$ -critical, nonlienar Schrödinger equation when  $d \geq 3$* , arXiv:0912.2467v3.
- [9] B. Dodson, *Global well-posedness and scattering for the defocusing,  $L^2$ -critical, nonlienar Schrödinger equation when  $d = 2$* , arXiv:1006.1375.
- [10] B. Dodson, *Global well-posedness and scattering for the defocusing,  $L^2$ -critical, nonlienar Schrödinger equation when  $d = 1$* , arXiv:1010.0040.
- [11] B. Dodson, *Global well-posedness and scattering for the mass critical nonlienar Schrödinger equation with mass below the mass of the ground state*, arXiv:1104.1114.
- [12] T. Duyckaerts and F. Merle, *Dynamic of threshold solutions for energy-critical NLS*. GAFA., **18:6**(2009), 1787–1840.

- [13] T. Duyckaerts and S. Roudenko, *Threshold solutions for the focusing 3D cubic Schrödinger equation*, Revista. Math. Iber., **26:1**(2010), 1–56.
- [14] D. Foschi, *Inhomogeneous Strichartz estimates*, J. Hyper. Diff. Equat., **2**(2005), 1–24.
- [15] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equation*, J. Math. Pures Appl., **64**(1985), 363–401.
- [16] J. Ginibre and G. Velo, *Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equation*, Ann. Inst. Henri Poincaré, Physique théorique, **43:4**(1985), 399–442.
- [17] S. Ibrahim, N. Masmoudi and K. Nakanishi, *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, to appear in Analysis & PDE.
- [18] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120:5**(1998), 955–980.
- [19] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case*, Invent. Math., **166**(2006), 645–675.
- [20] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation*, Acta Math., **201:2**(2008), 147–212.
- [21] S. Keraani, *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*, J. Diff. Equat., **175**(2001) 353–392.
- [22] S. Keraani, *On the blow up phenomenon of the critical Schrödinger equation*, J. Funct. Anal., **265**(2006), 171–192.
- [23] R. Killip, T. Tao and M. Visan, *The cubic nonlinear Schrödinger equation in two dimensions with radial data*, J. Eur. Math. Soc., **11:6**(2009), 1203–1258.
- [24] R. Killip and M. Visan, *The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher*, Amer. J. Math., **132:2**(2010), 361–424.
- [25] R. Killip, M. Visan and X. Zhang, *The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher*, Analysis & PDE. **1:2**(2008), 229–266.
- [26] D. Li and X. Zhang, *Dynamics for the energy critical nonlinear Schrödinger equation in high dimensions*, J. Funct. Anal., **256:6**(2009), 1928–1961.
- [27] C. Miao, Y. Wu and G. Xu, *Dynamics for the focusing, energy-critical nonlinear Hartree equation*, preprint.
- [28] C. Miao, G. Xu and L. Zhao, *Global well-posedness, scattering and blow-up for the energy-critical, focusing Hartree equation in the radial case*, Colloquium Mathematicum, **114**(2009), 213–236.
- [29] C. Miao, G. Xu and L. Zhao, *Global well-posedness and scattering for the mass-critical Hartree equation with radial data*, J. Math. Pures Appl., **91**(2009), 49–79.
- [30] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2*, J. Funct. Anal., **169**(1999), 201–225.
- [31] K. Nakanishi, *Remarks on the energy scattering for nonlinear Klein-Gordon and Schrödinger equations*, Tohoku Math. J., **53**(2001), 285–303.
- [32] K. Nakanishi and W. Schlag, *Global dynamics above the ground state energy for the cubic NLS equation in 3D*, to appear in Calc. of Variations and PDE.
- [33] T. Ogawa and Y. Tsutsumi, *Blow-up of  $H^1$  solution for the nonlinear Schrödinger equation*, J. Diff. Equat., **92**(1991), 317–330.
- [34] E. Ryckman and M. Visan, *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in  $\mathbb{R}^{1+4}$* , Amer. J. Math., **129**(2007), 1–60.

- [35] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura. Appl. **110**(1976), 353–372.
- [36] T. Tao, *Nonlinear dispersive equations, local and global analysis*, CBMS. Regional conference Series in Mathematics, **106**. Published for the Conference Board of the Mathematical Science, Washington, DC; by the American Mathematical Society Providence, RI, 2006.
- [37] T. Tao, M. Visan and X. Zhang, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. PDEs, **32**(2007), 1281–1343.
- [38] M. Visan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions*, Duke Math. J., **138:2**(2007), 281–374.
- [39] J. Zhang, *Sharp conditions of global existence for nonlinear Schrödinger and Klein-Gordon equations*, Nonlinear Anal. T. M. A., **48:2**(2002), 191–207.
- [40] X. Zhang, *On Cauchy problem of 3D energy critical Schrödinger equation with subcritical perturbations*, J. Diff. Equat., **230:2**(2006), 422–445.

CHANGXING MIAO:

INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS,  
P. O. BOX 8009, BEIJING, CHINA, 100088,  
*E-mail address:* miao\_changxing@iapcm.ac.cn

GUIXIANG XU

INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS,  
P. O. BOX 8009, BEIJING, CHINA, 100088,  
*E-mail address:* xu\_guixiang@iapcm.ac.cn

LIFENG ZHAO

UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA,  
HEFEI, CHINA,  
*E-mail address:* zhaolifengustc@yahoo.cn